COLORING GRAPHS

3 General Coloring Theorems

We start with some basic definitions.

DEFINITION 3.1: Let $G = (V, E)$ be a graph. Define the set of vertices $\Gamma(v)$ as the vertices adjacent to v and the degree $d_G(v)$ as $|\Gamma(v)|$ (for $v \in V$). Let $\delta(G)$ and $\Delta(G)$ be the minimum and maximum degree over all vertices v in V, respectively. If $\delta(G) = \Delta(G) = k$, then G is said to be k-regular. The Szekeres-Wilf number $sw(G)$ is defined as $\max_H \delta(H)$, where the maximum is taken over all (spanned) subgraphs H of G.

Clearly, $\delta(G) \leq sw(G) \leq \Delta(G)$.

EXERCISES 3.1: On the Szekeres-Wilf number:

1. Give a graph G_1, G_2 and G_3 for which $\delta(G) < sw(G) < \Delta(G)$, $\delta(G) = sw(G) < \Delta(G)$. and $\delta(G) < sw(G) = \Delta(G)$, respectively.

2. Show that if G is connected and $d_G(x) < \Delta(G)$ for some $x \in V$, then $sw(G) < \Delta(G)$.

DEFINITION 3.2: Let $G = (V, E)$ be a graph. We state that c is a (proper) k-coloring of G if all the vertices in V are colored using k colors such that no two adjacent vertices have the same color. $\chi(G)$ is defined as the minimum k for which there exists a (proper) k-coloring of G.

THEOREM 3.1 (Szekeres-Wilf (1968)): $\chi(G) \leq sw(G) + 1$.

Proof: Define $H_n = G$ (where $|G| = n$) and choose $x_n \in H_n$ such that $d_{H_n}(x_n) = \delta(H_n)$. Next, define $H_{n-1} = H_n - \{x_n\}$. Similarly, for $i = n - 1, \ldots, 1$, select $x_i \in H_i$ such that $d_{H_i}(x_i) = \delta(H_i)$ and define $H_{i-1} = H_i - \{x_i\}$.

(Thus, H_i is the subgraph of G induced by the vertices $\{x_1, \dots, x_i\}$, for $i = 1, \dots, n$).
Using this construction we can color G using at most $m+1 = \max_{i=1}^{n} \delta(H_i) + 1 \leq sw(G) + 1$ colors as follows (we color x_i after x_{i-1}). Assign an arbitrary color to x_1. While coloring x_i, for $i > 1$, we know that, so far, at most m vertices adjacent to x_i have been colored (because $d_{H_i}(x_i) \leq m$), meaning that we have at least one color left for x_i.
Q.E.D.

As a direct consequence we find that $\chi(G) \leq \Delta(G) + 1$ (because $sw(G) \leq \Delta(G)$). To compute $sw(G)$ one can use the following algorithm:

ALGORITHM 3.1: Set $m = \delta(G), n = |G|$ and $H_n = G$. Next, **for** $(i = n, n-1, \dots, 1)$ $\{$ choose $x_i \in H_i$ such that $d_{H_i}(x_i) = \delta(H_i)$, set $m = \max(m, d_{H_i}(x_i))$ and $H_{i-1} = H_i - \{x_i\}$ $\}$. Then $m = sw(G)$.

Proof: Clearly, $m \leq sw(G)$ as $H_i \subset G$. Now, suppose $m < sw(G)$ and let H' be a subgraph of G such that $sw(G) = \delta(H') > m$. Let j be maximal such that $x_j \in H'$. Then, $d_{H_j}(x_j) = \delta(H_j) \leq m < sw(G) = \delta(H') \leq d_{H'}(x_j)$, however, $H' \subset H_j$, implying that $d_{H_j}(x_j) \geq \delta_{H'}(x_j)$.
Q.E.D.

EXERCISES 3.2: On the Szekeres-Wilf Theorem:

1. Give a graph G, with $|G| = n$, for which $\chi(G) = \Delta(G) + 1$.
2. Give a graph G for which $\chi(G) < \delta(G) \leq sw(G)$.

Next, we prove that complete graphs are essentially the only graphs for which $\chi(G) = \Delta(G)+1$. Instead of presenting the original proof by Brooks, we give another easier proof by Melnikov and Vizing (1969). It illustrates the device of switching colors in a subgraph spanned by the vertices of certain colors. Suppose that c is a (proper) k-coloring of a graph H and let L be a component of the subgraph H' spanned by the vertices of color $1, \dots, l$ (notice, this subgraph H' is not necessarily connected; therefore, it might consist of different components). Then, keeping the colors in the graph spanned by the vertices $V(H-L)$ and permuting the colors in L, results in another (proper) k-coloring $\tilde{c}$ of H.

THEOREM 3.2 (Brooks (1941)): Let $\Delta \geq 3, \Delta(G) \leq \Delta$ and $K_{\Delta+1} \not\subset G$, then $\chi(G) \leq \Delta$.

Proof: Assume that the theorem fails and let G be a graph of minimal order showing this. Pick $x \in V$ arbitrary and denote $\Gamma(x) = \{x_1, \ldots, x_d\}, d \leq \Delta$. G is a minimal counterexample, thus, there exists a Δ-coloring c of $H = G - \{x\}$ (the somewhat odd formulation of this theorem is needed to apply the theorem to H, as $\Delta(H)$ might be less than $\Delta(G)$). If some color i is not used by c to color one of the vertices in $\Gamma(x)$, we could use this color for x to obtain a Δ-coloring of G. This shows the following fact:
(i) $d = \Delta$ and all the nodes in $\Gamma(x)$ have a different color for any Δ-coloring of H. We say that x_i is colored with color i. Denote H_{ij} as the subgraph induced by the vertices in H that are colored i and j.
(ii) The vertices x_i and x_j have to belong to the same component C_{ij} of H_{ij}. Indeed, otherwise we could switch the colors i and j in the component of x_j to obtain a Δ-coloring of H where x_i and x_j both have color i contradicting (i).
(iii) The component C_{ij} is a path from x_i to x_j. First, x_i has exactly 1 adjacent vertex in H colored j. Let us explain: $|\Gamma_H(x_i)| \leq \Delta - 1$ and if two of these neighbors have the same color j then at least one other color, say $k \neq i$, is not used by $\Gamma_H(x_i)$. Therefore, coloring x_i with color k would result in a Δ-coloring of H where both x_i and x_k have the same color contradicting (i). There is at least one vertex in $\Gamma_H(x_i)$ with color j because x_i is connected to x_j in H_{ij}. Second, C_{ij} cannot contain a vertex y with a degree at least 3 in C_{ij} (notice, this y is colored i or j). Indeed, let y be the first such vertex on a path from x_i to x_j in C_{ij}. Now, $|\Gamma_H(y)| \leq \Delta$, thus, if 3 nodes in $\Gamma_H(y)$ have the same color, then at most $\Delta - 2$ colors are used to color $\Gamma_H(y)$, meaning that we can use a color k different from i and j for y to obtain a Δ-coloring of H where x_i and x_j belong to different components in H_{ij} contradicting (ii).

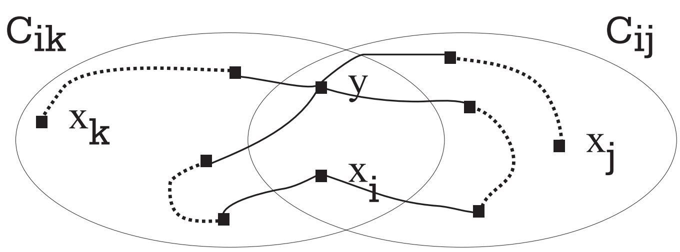

Fig. 2: Item (iv): $C_{ij} \cap C_{ik} \neq \{x_i\}$.

(iv) $C_{ij} \cap C_{ik} = \{x_i\}$. Suppose that $y \neq x_i$ is also part of the intersection (and thus colored i). Then, y has two neighbors colored i and two colored k (see Figure 2), meaning that at most $\Delta - 2$ colors are used to color $\Gamma_H(y)$. Therefore, we would recolor y by another color, say l, contradicting (ii).
So far, we did not use the fact that $K_{\Delta+1} \not\subset G$. This fact implies that one of the paths between x_i and x_j in C_{ij} has a length of at least 2 (actually, 3), say the path between x_1 and x_2. Thus, C_{12} contains a $y \neq x_2$ colored 2 that is adjacent to x_1. Now, switch the colors 1 and 3 in C_{13}, that is, the path between x_1 and x_3, to obtain a new Δ-coloring c' of H. Denote C'_{ij} as the paths in this new coloring c'. y is adjacent to x_1 colored 3 by c', meaning $y \in C'_{23}$, while y is also connected to x_2 by a path in C'_{12}, meaning $y \in C'_{23} \cap C'_{12}$ with $y \neq x_2$ contradicting (iv).

Q.E.D.

Notice, if $\Delta(G) < \Delta$, then Brooks' theorem does not provide us with any new information as $\chi(G)$ is always smaller than or equal to $\Delta(G) + 1$. For $\Delta = \Delta(G)$, it actually states that any graph for which $\chi(G) = \Delta(G) + 1$, with $\Delta(G) \geq 3$, must contain a $K_{\Delta(G)+1}$ as a subgraph.

Exercises 3.3: On Brooks' theorem:

1. Construct a graph G such that $\Delta(G) = 2$, $K_3 \not\subset G$ and $\chi(k) = \Delta(G) + 1 = 3$.

2. Show that $\chi(G) = 3$ for the Petersen graph G.

Definition 3.3: An orientation of a graph $G = (V, E)$ is a way of directing the edges of G. Thus, with an orientation G becomes a directed graph $\vec{G} = (V, \vec{E})$. Let $\{x_1, \ldots, x_n\}$ be a set of vertices in V, then $x_1 x_2 \ldots x_n$ is said to be a directed path of length n in $\vec{G}$ if $\overrightarrow{x_i x_{i+1}} \in \vec{E}$, for $i = 1, \ldots, n-1$. A directed cycle is defined as a directed path for which $x_1 = x_n$. Finally, let $l(\vec{G})$ be the maximum length of a directed path in $\vec{G}$.

Theorem 3.3 (Gallai, Roy (1967)): $\chi(G) = \min_{\rightarrow} l(\vec{G})$, where the minimum is taken over all the orientations of G.

Proof: (A) Given a k-coloring c with colors $\{1, \ldots, k\}$ of G. Direct an edge $xy \in E$ from x to y if the color of x is smaller than the color of y. Clearly, $l(\vec{G}) \leq k$ for this orientation $\vec{G}$. Thus, $\min_{\rightarrow} l(\vec{G}) \leq \chi(G)$.

(B) Let $\vec{G}$ be an orientation of G. We will show that G is $l(\vec{G})$-colorable. Let $\vec{E}_0$ be a minimal subset of $\vec{E}$ such that $\vec{H} = (V, \vec{E} - \vec{E}_0)$ is a directed graph that does not contain a directed cycle. Next, define $c(x)$ as the maximum length of all the directed paths in $\vec{H}$ that start in $x \in V$. Then, $c(x) \leq l(\vec{G})$. Now, color x with color $c(x)$. Let us check that this is a (proper) coloring of G. If $\vec{xy} \in \vec{E} - \vec{E}_0$ then $c(x) \geq c(y) + 1$, thus x and y have different colors. If $\vec{xy} \in \vec{E}_0$ then there exists a directed path of some length $p \geq 1$ from y to x in $\vec{H}$ (because the set $\vec{E}_0$ was minimal, therefore, adding the edge $\vec{xy}$ to $\vec{H}$ would result in a directed cycle). Hence, $c(y) \geq c(x) + p$, meaning that x and y were assigned a different color. Q.E.D.

Exercises 3.4: On coloring graphs:

Let G be a graph with n vertices $x_1, \ldots, x_n$ such that $d_1 \geq d_2 \geq \ldots \geq d_{n-1} \geq d_n$, where $d_i = d_G(x_i)$. Define the degree-sequence-number $dsn(G)$ of G as

$$dsn(G) = \max_{i=1}^{n} \{\min(d_i + 1, i)\}.$$

Show that the following equalities are valid:

1. $\delta(G) + 1 \leq dsn(G) \leq \Delta(G) + 1$,
2. $\chi(G) \leq dsn(G)$, without making use of 4.,
3. Draw a graph G such that $sw(G) + 1 \neq dsn(G)$,
4. $sw(G) + 1 \leq dsn(G)$.

Summary 3.1: In this section we have shown that the number of colors needed to color a graph $\chi(G)$ is upper bounded by $sw(G) + 1 \leq \Delta(G) + 1$. This upper bound is exact in some particular cases, but can be far from accurate (e.g., $\chi(G)$ can be smaller than $\delta(G)$, while $\delta(G) \leq sw(G)$). If G is connected and not regular, i.e., $\delta(G) \neq \Delta(G)$, then $\chi(G) \neq \Delta(G) + 1$ (by Exercise 3.1). Moreover, by Brooks' theorem we know that any graph for which $\chi(G) = \Delta(G) + 1$, with $\Delta(G) \geq 3$, must contain a $K_{\Delta(G)+1}$ as a subgraph. Thus, in essence, the only graphs that need $\Delta(G) + 1$ colors are complete graphs (as the only connected graph G that contains a $K_{\Delta(G)+1}$ is $K_{\Delta(G)+1}$ itself).

4 Coloring Planar Graphs

DEFINITION 4.1: A graph $G = (V, E)$ is said to be planar if G can be drawn in the plane such that no two edges intersect (except at a common vertex $v \in V$).

THEOREM 4.1 (Euler): Let n, e and f denote the number of vertices, edges and areas of a planar connected graph G. Then, $n - e + f = 2$.

Proof: We prove the property by induction on $e = |E|$. For $e = 1$, we have $n = 2$ and $f = 1$. For $e > 1$ we distinguish two cases: (i) Suppose there are no cycles in G (G is a tree), then $f = 1$ being the infinite area and $n = e + 1$. (ii) Otherwise, there exists some edge vw ($v, w \in V$) that is a border between 2 different areas (being any edge from an arbitrary cycle). Let $G' = (V, E - \{vw\})$, then G' is still planar and connected (otherwise vw did not separate 2 different areas). Thus, by induction, $n' - e' + f' = 2$. with $n = n'$, $e' = e - 1$ and $f' = f - 1$, implying $n - e + f = 2$.
Q.E.D.

LEMMA 4.1: Let $G = (V, E)$ be a connected planar graph, then $e \leq 3n - 6$ for $n > 2$.

Proof: This lemma is trivial for $G = P_3$. For $G \neq P_3$, all areas of G are bounded by 3 or more edges. Denote $\{a_1, a_2, \ldots, a_f\}$ as the set of all areas and $b(a_i)$ as the number of edges that bound area a_i, thus $b(a_i) \geq 3$ for all $i \in \{1, \ldots, f\}$. Hence, $3f \leq \sum_{i=1}^{f} b(a_i) \leq 2e$, as an edge bounds either one or two areas. Using Euler's theorem we find $6 = 3(n-e+f) \leq 3n-3e+2e = 3n - e$, which proves the lemma.
Q.E.D.

LEMMA 4.2: Let $G = (V, E)$ be a connected planar graph. Then, there exists a $v \in V$ such that $d_G(v) \leq 5$.

Proof: The average degree $\bar{d}$ of all vertices $v \in V$ equals $1/n \sum_{v \in V} d_G(v) = 2e/n$. By the previous lemma we have $\bar{d} \leq (6n - 12)/n = 6 - 12/n < 6$. Thus, there must be a vertex with a degree of at most five.
Q.E.D.

EXERCISES 4.1: On planar graphs:

1. Construct the smallest possible connected planar graph G such that $\delta(G) = 5$. Prove that it is minimal using Lemma 4.1.
2. Prove that $\chi(G) \leq 6$ for all planar graphs G.
3. Prove that for any connected planar graph G for which $K_3 \not\subset G$ and $n > 2$: $e \leq 2n - 4$.
4. Show that $K_{3,3}$ and K_5 are not planar.

THEOREM 4.2 (Kempe (1879)): Let $G = (V, E)$ be a planar graph, then $\chi(G) \leq 5$.

Proof: Assume that the theorem fails and let G be a graph of minimal order showing this. Let $x \in V$ be an vertex such that $d_G(x) \leq 5$ and denote $\Gamma(x) = \{x_1, \ldots, x_d\}, d \leq 5$. G is a minimal counterexample, thus, there exists a 5-coloring c of $H = G - \{x\}$. If some color $i \in \{1, \ldots, 5\}$ is not used by c to color one of the vertices in $\Gamma(x)$, we could use this color for x to obtain a 5-coloring of G. This shows that $d = 5$ and all the nodes in $\Gamma(x)$ have a different color for any 5-coloring of H. We say that x_i is colored with color i, for $i = 1, \ldots, 5$. Denote H_{ij} as the subgraph induced by the vertices in H that are colored i and j.
The vertices x_i and x_j have to belong to the same component[2] C_{ij} of H_{ij}. Indeed, otherwise we could switch the colors i and j in the component of x_j to obtain a 5-coloring of H where x_i and x_j both have color i. This implies that there is a path P_{13} from x_1 to x_3, the vertices of which are colored $1, 3, 1, \ldots, 3$ and a path P_{24} from x_2 to x_4, the vertices of which are colored $2, 4, \ldots, 4$. Now, $xP_{13}x$ forms a cycle C that separates x_2 from x_4 and each vertex in this cycle has a color different from different from 2 and 4. Consequently, this cycle C intersects P_{24}, which is impossible.

Q.E.D.

In 1976 Appel, Haken and Koch finally resolved Guthrie's conjecture (1852) that every planar graph is colorable by 4 colors, called the four color problem.

THEOREM 4.3 (Appel, Haken, Koch (1976)): Let $G = (V, E)$ be a planar graph, then $\chi(G) \leq 4$.

[2] Such components are called *ij Kempe chains*

The proof is very elaborate and computer-aided. A brief discussion is included to give some insight on how the problem got resolved. It is easy to show that the following theorem is equivalent to the four color theorem.

THEOREM 4.4: To proof the 4-color theorem it suffices to show that any triangulation is 4 colorable. A *triangulation* is a planar connected graph, for which every area (including the infinite) is bounded by exactly 3 edges.

Proof: Let G be an arbitrary planar graph. Suppose G has an area that is bounded by $b \geq 4$ edges. Then, there must be 2 vertices part of this boundary that are not adjacent to each other (otherwise we could add a vertex v in the middle of this area to obtain a planar representation of K_5). Thus, we can always add a new edge to G that produces two areas with less than b edges. By repeatedly applying this method we obtain a triangulation $\bar{G}$, called the *standardization* of G. Any 4-coloring of $\bar{G}$ clearly suffices to color G. Q.E.D.

If a minimal counter-example G_{min} were to exist, then G_{min} has to be a triangulation. Suppose G_{min} had an area bounded by $b \geq 4$ edges. Again, two vertices v_1, v_2 part of this boundary exist that are not connected with each other. By identifying[3] these vertices with each other, we obtain a planar graph G'_{min} that has fewer vertices. Hence, G'_{min} is colorable by 4 colors, but then so is G_{min} by assigning the color of w to both v_1 and v_2.

A *configuration* C consists of a subgraph with specification of its vertex degrees and the manner in which it is embedded in a planar triangulation. Examples are: (i) a vertex of degree 4, (ii) a *separating* circuit Q of length n (This is a circuit Q that separates G into two nonempty components when removed.), (iii) a degree 5 vertex with 3 degree 5 neighbors, etc. Such a configuration is called *reducible* if for any graph G containing C, one can make a graph G' with $|G'| < |G|$, such that any 4-coloring of G' produces a 4-coloring of G. The argument that we presented to demonstrate that a minimal counter-example has to be a triangulation actually shows that a *nonseparating* circuit of length > 3 is a reducible configuration. A proof similar to the 5-color theorem shows that a degree 4 vertex is also a reducible configuration.

[3] Meaning that we replace both vertices v_1 and v_2 by a single one called w, the neighbors of which are the neighbors of v_1 and v_2.

Apart from developing an algorithm that could check whether a (not too large) configuration is reducible. Appel, Haken and Koch made use of a discharging algorithm to find a set L of unavoidable configurations, such that any triangulation has to contain at least one of the configurations in L. The idea of discharging and unavoidable sets is due to Heesch and works as follows. We start by charging a triangulation G, by adding a charge equal to $6 - d_G(v)$ to every vertex v. Using Euler's equality and the fact that $3f = 2e$ for a triangulation G, we find that the total charge on G equals 12. Next, one can define an algorithm that exchanges the charges between the vertices. For instance, any degree 8 (or more) vertex distributes its (negative) charge equally between its degree 5 neighbors. If one can now show that this algorithm makes the charge of every vertex v nonpositive provided that a specific set L of configurations does not appear in G. Then, L is an unavoidable set of configuration as the total amount of charge always remains equal to 12, no matter what the discharging algorithm may be.
The technique used by Appel, Haken and Koch existed in finding a discharging algorithm that produced an unavoidable set of configurations L such that each element of L can be show to be reducible (by means of their algorithm). Whenever they encountered an element of L that could not be shown to be reducible, they adapted their discharging algorithm such that this configuration did not appear in the set L. Eventually they found a fairly complicated discharging algorithm that produced a set of 1936 unavoidable configurations that were all reducible (or contained a subcomponent that was reducible).

EXERCISES 4.2: On coloring planar graphs:

1. *The empire problem:* Given an integer r, what is the minimum number N_r of colors needed to color a map in which some countries have up to r colonies ? A country and its colonies should be colored the same, and no pair of colonies or of country and colony are adjacent. Show that $N_r \leq 6(r+1)$. [Hint: Let $G = D(M)$ be the dual graph of the map M and define G_r by identifying a country with its colonies. Use the fact that $n_H \leq (r+1)n_{H_r}$, $e_{H_r} \leq e_H$ and $e_H \leq 3n_H - 6$ for any subgraph H of G to find an expression for $\delta(H_r)$, where $H_r \subseteq G_r$ is obtained from H by identifying a country with its colonies.]