On the Asymptotic Insensitivity of the Supermarket Model in Processor Sharing Systems

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The supermarket model is a popular load balancing model where each incoming job is assigned to a server with the least number of jobs among d randomly selected servers. Several authors have shown that the large scale limit in case of processor sharing servers has a unique insensitive fixed point, which naturally leads to the belief that the queue length distribution in such a system is insensitive to the job size distribution as the number of servers tends to infinity. Simulation results that support this belief have also been reported. However, global attraction of the unique fixed point of the large scale limit was not proven except for exponential job sizes, which is needed to formally prove asymptotic insensitivity. The difficulty lies in the fact that with processor sharing servers, the limiting system is in general not monotone.

In this paper we focus on the class of hyperexponential distributions of order 2 and demonstrate that for this class of distributions global attraction of the unique fixed point can still be established using monotonicity by picking a suitable state space and partial order. This allows us to formally show that we have asymptotic insensitivity within this class of job size distributions. We further demonstrate that our result can be leveraged to prove asymptotic insensitivity within this class of distributions for other load balancing systems.

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1 INTRODUCTION

The supermarket model refers to a popular load balancing model consisting of N homogeneous servers and Poisson arrivals with rate λN, where for each incoming job d servers are selected at random and the job joins the queue with the fewest jobs. In the seminal papers [14, 19] it was shown that when job sizes are exponential with mean 1 the probability of having k or more jobs in a server converges to $\lambda^{(d^k-1)/(d-1)}$ as the number of servers N tends to infinity. Hence the queue length decays doubly exponential as soon as $d > 1$, demonstrating the power of having d choices. While the authors considered First-Come-First-Served (FCFS) servers, the result also applies to processor sharing (PS) servers as both systems are equivalent when the job sizes are exponential.

A modularized program to study the supermarket model with non-exponential job sizes was proposed in [1] for both FCFS and PS servers. The program relies on an ansatz that asserts that, for a randomized load balancing scheme *in equilibrium*, any fixed number of queues become independent of one another as the number of servers tends to infinity. Using this ansatz hypothesis the limiting steady-state queue length distribution and other performance measures of interest can be computed by studying the queue at the cavity (e.g., [8, 9]).

For the supermarket model with processor sharing the steady-state queue length distribution of the queue at the cavity is that of an M/G/1/PS queue with arrival rates that depend on the queue

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length. As the queue length distribution of such a queue is known to be insensitive to the job size distribution (meaning only the mean job size matters) [3], this naturally leads to the conjecture that the queue length distribution becomes insensitive to the job size distribution as the number of servers tends to infinity, which we refer to as *asymptotic insensitivity*. In [1] the authors noted that for the supermarket model with a finite fixed number of servers N with PS service, the queue length distribution is not insensitive, meaning only as N tends to infinity the sensitivity vanishes. In addition, the authors showed that the cavity map that is used to compute the limiting steady-state queue length distribution in case of PS servers has a unique fixed point that corresponds to the same distribution as in the exponential case, yielding further support for the conjecture of asymptotic insensitivity. While the ansatz was proven in [2, 15] for various load balancing policies and proving the ansatz for the supermarket model with PS servers would settle the conjecture, it is still an open problem.

In [17] the authors also considered the supermarket model with PS service and general service times. The authors used measure-valued processes and martingale techniques to show that the limit of the empirical distributions satisfies a set of partial differential equations (PDEs). These PDEs correspond to the transient behavior of an M/G/1/PS queue with a queue length and *time* dependent arrival rate. The authors further showed that this set of PDEs has a unique fixed point, which is in agreement with the result in [1]. However, as stated after listing their main contributions, in order to prove asymptotic insensitivity of the limit of the stationary measures, global attraction of the fixed point must be proven. Instead of providing such a proof, the authors present simulation results supporting asymptotic insensitivity.

A major challenge in proving asymptotic insensitivity for the the supermarket model with PS servers lies in overcoming the apparent lack of monotonicity in such systems. In this paper we show that monotonicity arguments can still be leveraged if we restrict ourselves to the class of hyperexponential distributions of order 2. More specifically, we prove that the limiting steady-state queue length distribution of the supermarket model with PS servers and order 2 hyperexponential job sizes is the same as the limiting distribution for exponential job sizes. In other words we prove *asymptotic insensitivity of the limiting steady-state queue length distribution within the class of hyperexponential distributions of order* 2. The class of hyperexponential distributions of order 2 is often used in performance modeling as it can be regarded as a mixture of long and short jobs and can be used to match any squared coefficient of variation (SCV) larger than one.

It is worth noting that convergence of the steady state measures has been established in some specific cases even for systems that are not monotone. For instance, in [12] it is shown that various load balancing policies for FCFS servers achieve vanishing delays in the heavy traffic regime when the load equals $1 - N^{-\alpha}$, for $0 < \alpha < 0.5$, when the job sizes have general order-2 Coxian distributions. The supermarket model is one of the policies considered in [12], but in this case d scales as $O(N^{\alpha} \log(N))$, whereas in this paper d is a constant independent of N, which implies that delays do not vanish, and servers use PS instead of FCFS.

The approach taken in this paper is as follows. For job sizes with an order 2 hyperexponential distribution, the sample paths of the stochastic process of the supermarket model consisting of N servers converge to the solution of a set of ordinary differential equations (ODEs) that is shown to have a unique fixed point. The main step to establish asymptotic insensitivity then exists in showing that the fixed point of the set of ODEs is a global attractor. We show that the set of ODEs is monotone by using a Coxian representation of the hyperexponential distribution and defining a suitable state space and partial order, from which global attraction follows without much effort. The work in this paper is in this regard somewhat similar to [16], where a Coxian representation and a suitable state space and partial order was also used to prove global attraction of some load balancing systems. However the systems considered in [16] are restricted to FCFS servers, which simplifies the set of ODEs and especially the proof that the set of ODEs is monotone. Moreover, the result in [16] applies to hyperexponential distributions of any order, while for PS servers the approach appears to be limited to order 2 distributions (see Section 7). Similar to [16], we also present our global attraction result in such a manner that it can be used to prove global attraction for models with PS servers other than the supermarket model and we demonstrate this for the traditional push strategy (see Section 8). To avoid some technical issues, we assume that the buffer size at each server is finite.

The main contributions of the paper are as follows:

- We prove asymptotic insensitivity of the limiting steady-state queue length distribution within the class of hyperexponential distributions of order 2 for the supermarket model with PS servers.
- We present our global attraction result used to prove asymptotic insensitivity in such a manner that it may also be leveraged for other load balancing policies with PS servers and demonstrate this using the traditional push algorithm.

The paper is structured as follows. In Section 2 we discuss the model under consideration and present the Coxian representation. In Section 3 we introduce the set of ODEs describing the mean field limit, while the state space and partial order are presented in Section 4. Our global attraction result is stated and proven in Section 5. In Section 6 we show that the assumptions of our global attraction result are satisfied for the supermarket model and prove that the set of ODEs has a unique fixed point that corresponds to the same queue length distribution as in the exponential case. The asymptotic insensitivity result is presented in Section 7, while in Section 8 we demonstrate that our results are not limited to the supermarket model. Finally conclusions are drawn in Section 9.

2 MODEL DESCRIPTION

We focus on the supermarket model, also known as the JSQ(d) load balancing policy, with processor sharing servers. In this model, arrivals occur according to a Poisson process with rate λN, we have a set of N servers that use processor sharing and each incoming job is immediately assigned to a server by selecting a server with the least number of jobs among a set of d random servers (with ties being broken at random). We assume the processing speed of a server equals 1 and when n jobs are present in a server, each job receives an equal share $1/n$ of the processing speed of the server. We further assume each server has a finite buffer of size B, meaning an incoming job is lost if d servers with a full buffer are selected. The finiteness of the buffer allows us to avoid certain technical issues and is not an uncommon assumption in mean field modeling [6, 7]. Further in a real system all buffers are finite and there is hardly any difference between having a huge finite buffer or an infinite buffer (as long as the system is stable).

We consider order 2 hyperexponential job sizes with a mean equal to 1, meaning $\lambda < 1$ suffices for the system to be stable. More specifically, with some probability $\tilde{p}$ jobs have an exponential size with mean $1/\mu_1$ and with probability $1-\tilde{p}$ jobs have an exponential size with mean $1/\mu_2$ with $\mu_1 > \mu_2$, such that $\tilde{p}/\mu_1 + (1-\tilde{p})/\mu_2 = 1$. While the standard phase-type representation $(\tilde{\alpha}, \tilde{S})$ of a hyperexponential distribution is given by $\tilde{\alpha} = (\tilde{p}, 1-\tilde{p})$ and

$$\tilde{S} = \begin{bmatrix} -\mu_1 & 0 \\ 0 & -\mu_2 \end{bmatrix},$$

hyperexponential distributions also have a Coxian representation, see [16, Proposition 1], which in case of 2 phases corresponds to a representation (α, S) with $\alpha = (1, 0)$ and

$$S = \begin{bmatrix} -\mu_1 & p_1\mu_1 \\ 0 & -\mu_2 \end{bmatrix},$$

where $p_1 = (1-\tilde{p})(1-\mu_2/\mu_1)$ and $(1-p_1)\mu_1 > \mu_2$. In fact one can readily check (by computing the Laplace Stieljes transform) that any distribution with an order 2 Coxian representation and $(1-p_1)\mu_1 > \mu_2$, is a hyperexponential distribution with $\tilde{p} = 1 - p_1\mu_1/(\mu_1-\mu_2)$. For further use we denote $\nu_1 = \mu_1(1-p_1)$ and $\nu_2 = \mu_2$. To establish monotonicity we formulate the mean field limit using the Coxian representation (α, S).

The main challenge to prove global attraction using monotonicity arguments, is to pick a set of variables that capture the system state such that the set of ODEs that describes the dynamics of the mean field model in terms of these variables is monotone with respect to some partial order on the associated state space. When the job sizes are exponential, [19] showed that the set of ODEs given by

$$\frac{d}{dt}h_j(t) = \lambda(h_{j-1}^d(t) - h_j^d(t)) - (h_j(t) - h_{j+1}(t)),$$

where the variables $h_j(t)$ represent the fraction of the servers with j or more jobs at time t is monotone with respect to the pointwise partial order.

When the servers are FCFS servers as in [16] and the job sizes are hyperexponential of order 2, then it suffices to use a set of variables that represent the fraction of servers with j or more jobs (denoted as $h_{j,1}$ in [16]) and a set of variables for the fraction of servers with j or more jobs for which the server is in phase 2 (denoted as $h_{j,2}$ in [16]). In this case a stronger partial order $\leq_C$ is required to get a monotone system. This order is such that $h \leq_C \tilde{h}$ if

$$h_{j_1,1} - h_{j_1,2} + h_{j_2,2} \leq \tilde{h}_{j_1,1} - \tilde{h}_{j_1,2} + \tilde{h}_{j_2,2},$$

for all $j_1 \geq j_2 \geq 1$. Note that $h_{j_1,1} - h_{j_1,2} + h_{j_2,2}$ is the fraction of servers with j_1 or more jobs in service phase 1 (given by $h_{j_1,1} - h_{j_1,2}$) plus the fraction of servers with j_2 or more jobs in service phase 2 (given by $h_{j_2,2}$).

For PS servers the system state is clearly more complex as we need to keep track of the number of jobs in service phase 1 and phase 2. Therefore a more complex set of variables denoted as $h_{i,j}$ is required, where $h_{i,j}$ represents the fraction of servers with at least $i+j$ jobs of which at least j jobs are in phase 2, for $i, j \geq 0$. As a result, the partial order in case of PS servers is more involved as is the set of ODEs that describe the evolution of the mean field limit. This implies that proving monotonicity requires different arguments and is more challenging in case of PS servers. Indeed, the monotonicity proof in case of FCFS servers given in [16, Proposition 6] is fairly straightforward compared to the proof of Proposition 5.2 in this paper.

3 THE SET OF ODES

In this section we introduce the set of ODEs that describes the mean field limit when the servers use processor sharing, have a finite buffer of size B and the order 2 hyperexponential job sizes are represented in Coxian form. Let $h_{i,j}(t)$, for $i, j \geq 0$, denote the fraction of servers with at least $i+j$ jobs of which at least j jobs are in service phase 2 at time $t \geq 0$. We set $h_{0,0}(t) = 1$ and $h_{i,j}(t) = 0$, if $i + j > B$. We define $1[P]$ to be 1 if the property P holds and 0 otherwise. To ease the presentation we define

- $y_{i,j}(h(t)) = h_{i,j}(t) - h_{i-1,j+1}(t)$, for $i > 0, j \geq 0$,
- $y_{0,j}(h(t)) = h_{0,j}(t) - h_{0,j+1}(t)$, for $j \geq 0$,
- $w_{i,j}(h(t)) = y_{i,j}(h(t)) - y_{i+1,j}(h(t))$, for $i, j \geq 0$,

Note that $y_{i,j}(h(t))$ represents the fraction of the queues with at least $i+j$ jobs of which exactly j jobs are in service phase 2 at time $t \geq 0$, $w_{i,j}(h(t))$ is the fraction of the queues with exactly i jobs in service phase 1 and exactly j jobs in service phase 2 at time $t \geq 0$. An illustration of these

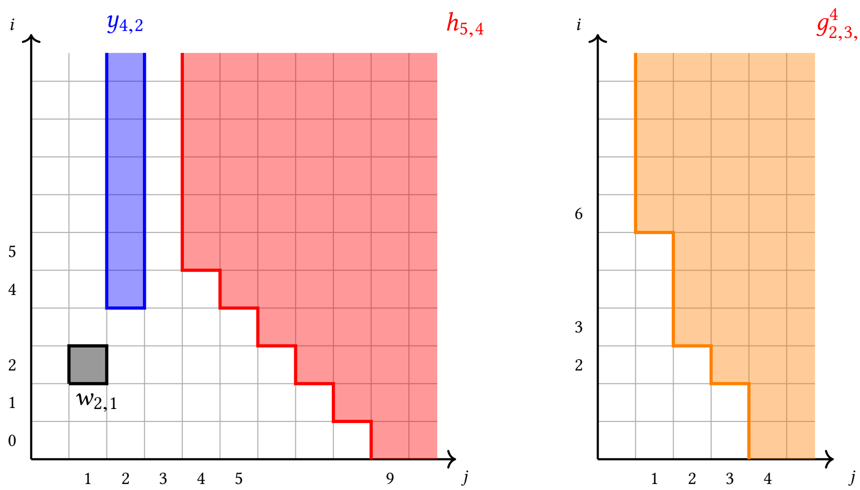

Fig. 1. Illustration of the variables $h_{i,j}, y_{i,j}(h) = h_{i,j} - h_{i-1,j+1}$ and $w_{i,j}(h) = y_{i,j}(h) - y_{i+1,j}(h)$ (left) and $g^{(j)}_{i_1,\dots,i_s}(h)$ (right).

variables can be seen in Figure 1(left). While the notation may appear a bit heavy, its usefulness becomes apparent in the next section.

In order to understand the set of ODEs that is presented next, we make the following observations:

- Phase changes increase $h_{i,j}(t)$ if and only if such a phase change happens in a server with exactly $j-1$ jobs in phase 2 and exactly $k+1$ jobs in phase 1, for $k \geq i$. This happens at rate $p_1\mu_1$ times the fraction $(k+1)/(k+j)$ of jobs that are in phase 1. This explains the appearance of the first sum in (1).
- Service completions decrease $h_{i,j}(t)$ if a service completion happens in a server with exactly $i+j$ jobs. These service completions occur at rate $(\nu_1(i-k)+\nu_2(j+k))/(i+j)$ if $i-k$ of the $i+j$ jobs are in phase 1, which explains the second sum in (1).
- Service completions also decrease $h_{i,j}(t)$ if a service completion of a job in phase 2 occurs in a server with $k+j > i+j$ jobs, of which exactly j jobs are in phase 2. These service completions occur at rate $\nu_2 j/(j+k)$ when there are $j+k$ jobs. This yields the third sum in (1).

Let $f_{i,j}(h(t))$ capture the changes due to other events, such as arrivals (specified later on), then the system of ODEs is given by:

$$\frac{d}{dt}h_{i,j}(t) = f_{i,j}(h(t)) + 1[j \geq 1]p_1\mu_1 \sum_{k=i}^{\infty} w_{k+1,j-1}(h(t))\frac{k+1}{k+j}$$
$$- \sum_{k=0}^{i} w_{i-k,j+k}(h(t))\frac{\nu_1(i-k)+\nu_2(j+k)}{i+j} - \nu_2 \sum_{k=i+1}^{\infty} w_{k,j}(h(t))\frac{j}{k+j} \tag{1}$$

for all $i, j \geq 0$ with $(i, j) \neq (0, 0)$ and $i + j \leq B$. Note that $w_{i,j}(h(t)) = 0$ whenever $i + j > B$.

Remark. *In this paper we show that this set of ODEs has a unique fixed point that is a global attractor using monotonicity arguments. The monotonicity is proven by separately showing that $f_{i,j}(h(t))$ is monotone and that all the remaining terms are monotone. This implies that our result can also be used to prove global attraction for other systems of ODEs as long as they have the above form and $f_{i,j}(h(t))$ is monotone, as demonstrated in Section 8.*

4 STATE SPACE AND PARTIAL ORDER

We define the state space Ω_B of the mean field model in terms of the variables $h_{i,j}$ as follows

$$\Omega_B = \Big\{(h_{i,j})_{i,j\geq 0,\ (i,j)\neq(0,0)} \Big| \ 0 \leq h_{i,j} \leq 1, h_{i,j} = 0 \text{ for } i+j > B, h_{i,j} \geq h_{i+1,j},$$
$$h_{i,j} \geq 1[i \geq 1] h_{i-1,j+1}, (h_{i+1,j} - h_{i,j+1}) - (h_{i+2,j} - h_{i+1,j+1}) \geq 0\Big\}. \tag{2}$$

The last condition states that $w_{i+1,j} \geq 0$, where $w_{i,j}$ and $y_{i,j}$ is defined analogue to $w_{i,j}(h(t))$ and $y_{i,j}(h(t))$, respectively. Note that from the last two conditions we get

$$h_{i,j} \geq h_{i,j+1}. \tag{3}$$

We now define the variables $g^j_{i_1,\ldots,i_s}(h)$ illustrated in Figure 1(right) that will be used to define the partial order.

Definiton 4.1. For $h \in \Omega_B$, $j \geq 1$, $0 \leq s \leq j$ and $0 = i_0 < i_1 < i_2 < \ldots < i_s$, we set

$$g^j_{i_1,\ldots,i_s}(h) = h_{0,j} + \sum_{k=1}^{s} y_{i_k,j-k}(h).$$

Definiton 4.2 (Partial order $\leq_C$ on Ω_B)**.** Let $h, \tilde{h} \in \Omega_B$. We state that $h \leq_C \tilde{h}$ if

$$g^j_{i_1,\ldots,i_s}(h) \leq g^j_{i_1,\ldots,i_s}(\tilde{h}) \tag{4}$$

for all $j \geq 1$, $0 \leq s \leq j$ and $0 = i_0 < i_1 < i_2 < \ldots < i_s$.

Remark 4.3. By noting that $g^{i+j}_{1,2,\ldots,i}(h) = h_{i,j}$, (4) implies that $h_{i,j} \leq \tilde{h}_{i,j}$ for all i, j.

Remark 4.4. To see why the pointwise partial order does not suffice, consider $h, \tilde{h} \in \Omega_B$ with $w_{0,1}(h) = w_{2,0}(h) = 1/2$ and $w_{1,1}(\tilde{h}) = w_{1,0}(\tilde{h}) = 1/2$. In other words, in state h half of the servers contain a single job in phase 2 and the other servers contain 2 jobs both in phase 1, while in state $\tilde{h}$ half of the servers contain a single job in phase 1 and the remaining servers contain a phase 1 and phase 2 job. It is easy to check that $h_{i,j} \leq \tilde{h}_{i,j}$ for all $i, j \geq 0$ (as $h_{1,0} = \tilde{h}_{1,0} = 1$, $h_{2,0} = \tilde{h}_{2,0} = 1/2$, $h_{0,1} = \tilde{h}_{0,1} = 1/2$, $h_{1,1} = 0$, $\tilde{h}_{1,1} = 1/2$ and $h_{i,j} = \tilde{h}_{i,j} = 0$ for all other i, j). Hence, h is smaller than $\tilde{h}$ in the pointwise order. However, idle servers are created at rate $\nu_2/2$ in state h and at rate $\nu_1/2$ in state $\tilde{h}$. As $\nu_1 > \nu_2$, this implies that $\tilde{h}_{1,0}$ decreases faster than $h_{1,0}$, meaning the system is not monotone with respect to the pointwise partial order. Note that $h \not\leq_C \tilde{h}$ as $g^1_2(h) = 1$ and $g^1_2(\tilde{h}) = 1/2$, so the fact that idle servers are created at a higher rate from state $\tilde{h}$ than state h does not violate monotonicity with respect to the order $\leq_C$.

Remark 4.5. Consider two sets $\mathcal{A}$ and $\tilde{\mathcal{A}}$ of N servers and let h and $\tilde{h}$ be their corresponding states in Ω_B. The intuition behind the order $\leq_C$ is that it should be such that $h \leq_C \tilde{h}$ implies that there exists a mapping $m : \mathcal{A} \to \tilde{\mathcal{A}}$ such that both the total number of jobs as well as the number of jobs in phase two for any server $a \in \mathcal{A}$ is dominated by the corresponding quantities of server $m(a) \in \tilde{\mathcal{A}}$. The above example shows that this is not the case for the pointwise order.

For further use we remark that for $j \geq s \geq 1$:

$$g^j_{i_1,\ldots,i_s}(h) = g^j_{i_1,\ldots,i_{s-1}}(h) + y_{i_s,j-s}(h) \tag{5}$$

and

$$w_{i_s,j-s}(h) = g^j_{i_1,\ldots,i_s}(h) - g^j_{i_1,\ldots,i_{s-1},i_s+1}(h), \tag{6}$$
$$w_{0,j}(h) = g^j(h) - g^{j+1}_1(h). \tag{7}$$

To simplify the notation we set $g^j_{i_1,\ldots,i_j,k}(h) = g^j_{i_1,\ldots,i_j}(h)$ if $k \geq 1$, i.e., additional indices after position j have no impact. Also note that for $i_s > B-(j-s)$ we have

$$g^j_{i_1,\ldots,i_{s-1},i_s}(h) = g^j_{i_1,\ldots,i_{s-1}}(h), \tag{8}$$

as $y_{i_s,j-s}(h) = 0$ for $i_s + j - s > B$. The next two Lemmas are used further on to prove monotonicity of the set of ODEs in (1) with respect to the order $\leq_C$.

Lemma 4.6. *Let $j, s, i_1, \ldots, i_s$ be as in Definition 4.2. Let $c_1, \ldots, c_s \in \mathbb{R}$ and $c_0 = 0$. We then have*

$$\sum_{k=1}^{s} w_{i_k,j-k}(h)c_k = g^j_{i_1,\ldots,i_s}(h)c_s - \sum_{k=0}^{s-1} g^j_{i_1,\ldots,i_k,i_{k+1}+1,\ldots,i_s+1}(h)(c_{k+1}-c_k). \tag{9}$$

Proof. First, repeatedly using (5) and (6) gives

$$\sum_{k=1}^{s} w_{i_k,j-k}(h) = g^j_{i_1,\ldots,i_s}(h) - g^j_{i_1+1,\ldots,i_s+1}(h),$$

which yields that the left hand side of (9) can be rewritten as:

$$g^j_{i_1,\ldots,i_s}(h)c_1 - g^j_{i_1+1,\ldots,i_s+1}(h)c_1 + \sum_{k=2}^{s} w_{i_k,j-k}(h)(c_k - c_1). \tag{10}$$

Applying (5) and (6) implies that (10) is equivalent to

$$\begin{aligned} & g^j_{i_1,\ldots,i_s}(h)c_1 - g^j_{i_1+1,\ldots,i_s+1}(h)c_1 - \sum_{k=2}^{s}(-g^j_{i_1,\ldots,i_k}(h) + g^j_{i_1,\ldots,i_{k-1},i_k+1}(h))(c_k - c_1) \\ &= g^j_{i_1,\ldots,i_s}(h)c_1 - g^j_{i_1+1,\ldots,i_s+1}(h)c_1 - \sum_{k=2}^{s}(-g^j_{i_1,\ldots,i_k,i_{k+1}+1,\ldots,i_s+1}(h) + g^j_{i_1,\ldots,i_{k-1},i_k+1,\ldots,i_s+1}(h))(c_k - c_1). \end{aligned} \tag{11}$$

Rearranging the terms in (11) (and noting that the sum can start in $k = 1$), we conclude that the left hand side of (9) can be written as

$$\begin{aligned} & g^j_{i_1,\ldots,i_s}(h)c_1 - g^j_{i_1+1,\ldots,i_s+1}(h)c_1 + g^j_{i_1,\ldots,i_s}(h)(c_s - c_1) - 1[s \geq 2]g^j_{i_1,i_2+1,\ldots,i_s+1}(h)(c_2 - c_1) \\ & - \sum_{k=2}^{s-1} g^j_{i_1,\ldots,i_k,i_{k+1}+1,\ldots,i_s+1}(h)(c_{k+1} - c_k) = g^j_{i_1,\ldots,i_s}(h)c_s - \sum_{k=0}^{s-1} g^j_{i_1,\ldots,i_k,i_{k+1}+1,\ldots,i_s+1}(h)(c_{k+1} - c_k). \end{aligned} \tag{12}$$

This finishes the proof. □

Lemma 4.7. *Let $j, s, i_1, \ldots, i_s$ be as in Definition 4.2. If $s > \tilde{s}$, then*

$$\frac{i_s}{i_s + j - s} > \frac{i_{\tilde{s}}}{i_{\tilde{s}} + j - \tilde{s}}. \tag{13}$$

Proof. We have that

$$\frac{i_{k+1}}{i_{k+1} + j - (k+1)} > \frac{i_k}{i_k + j - k} \tag{14}$$

is equivalent to

$$(i_{k+1} - i_k)(j - k) > -i_k, \tag{15}$$

which is true, as the right hand side is negative. The statement then follows by repeatedly using (14) $s - \tilde{s}$ times. □

5 GLOBAL ATTRACTION

In this section, we define empty summations to be 0. We now state three assumptions for $f_{i,j}(h)$ that suffice for the set of ODEs in (1) to have a global attractor in Ω_B. We prove that these assumptions hold for the supermarket model in Section 6.

Assumption 1. *The functions $f_{i,j}(h) : \Omega_B \to \mathbb{R}$ are such that for any $h_0 \in \Omega_B$, the set of ODEs given by (1) has a unique solution $h(t) : [0, \infty) \to \mathbb{R}$ with $h(0) = h_0$.*

Definiton 5.1. For $h \in \Omega_B$, $j \geq 1$, $0 \leq s \leq j$ and $0 = i_0 < i_1 < i_2 < \ldots < i_s$, we set

$$F^j_{i_1,\ldots,i_s}(h) = f_{0,j}(h) + \sum_{k=1}^{s} (f_{i_k,j-k}(h) - f_{i_k-1,j-k+1}(h)).$$

Assumption 2. *The functions $f_{i,j}(h) : \Omega_B \to \mathbb{R}$ are such that for all $j \geq 1$ and all i_k's as in (4) we have*

$$F^j_{i_1,\ldots,i_s}(h) \leq F^j_{i_1,\ldots,i_s}(\tilde{h}) \tag{16}$$

if $h \leq_C \tilde{h}$ and $g^j_{i_1,\ldots,i_s}(h) = g^j_{i_1,\ldots,i_s}(\tilde{h})$.

Assumption 3. *The functions $f_{i,j}(h) : \Omega_B \to \mathbb{R}$ are such that the set of ODEs given by (1) has a unique fixed point π in Ω_B.*

Under the first two assumptions we prove that the partial order $\leq_C$ is preserved over time.

Proposition 5.2. *Assume that Assumptions 1-2 hold and let $h_0, \tilde{h}_0 \in \Omega_B$. Let $h(t)$ and $\tilde{h}(t)$ be the unique solution of (1) with $h(0) = h_0$ and $\tilde{h}(0) = \tilde{h}_0$, respectively. If $\nu_1 = \mu_1(1 - p_1) > \mu_2 = \nu_2$ and $h_0 \leq_C \tilde{h}_0$ then $h(t) \leq_C \tilde{h}(t)$ for any $t \geq 0$.*

Proof. We show that (4) is retained over time. Suppose that $g^j_{i_1,\ldots,i_s}(h(t)) = g^j_{i_1,\ldots,i_s}(\tilde{h}(t))$ for some $j \geq 1$ and $0 < i_1 < \ldots < i_s$, with $s \leq j$. We need to show that

$$\frac{d}{dt} g^j_{i_1,\ldots,i_s}(h(t)) \leq \frac{d}{dt} g^j_{i_1,\ldots,i_s}(\tilde{h}(t)), \tag{17}$$

as otherwise (4) is violated at some time greater than t. Hence, it suffices to show that $\frac{d}{dt} g^j_{i_1,\ldots,i_s}(h(t))$ is non-decreasing in $g^{j'}_{i'_1,\ldots,i'_{s'}}(h(t))$ for all sets of indices $\{j', i'_1, \ldots, i'_{s'}\}$, as in (4), different from $\{j, i_1, \ldots, i_s\}$. Due to Assumption 2, this holds for the terms associated to $f_{i,j}(h(t))$ and it suffices to show that this also holds for the remaining terms corresponding to phase changes and service completions.

The drift of $g^j_{i_1,\ldots,i_s}(h(t))$ due to phase changes can be written as

$$1[j \geq 1] p_1 \mu_1 \left(\sum_{v=1}^{s} \sum_{k=i_{v-1}+1}^{i_v - 1} w_{k,j-v}(h(t)) \frac{k}{k+j-v} + 1[j \geq s+1] \sum_{k=i_s+1}^{\infty} w_{k,j-s-1}(h(t)) \frac{k}{k+j-(s+1)} \right), \tag{18}$$

as illustrated in Figure 2(left).

For ease of notation set $i_{s+1} = \infty$ and suppress the dependence on $h(t)$. Using (6), we have that (18) is equal to

$$1[j \geq 1] p_1 \mu_1 \left(\sum_{v=1}^{s} \sum_{k=i_{v-1}+1}^{i_v - 1} (g^j_{i_1,\ldots,i_{v-1},k} - g^j_{i_1,\ldots,i_{v-1},k+1}) \frac{k}{k+j-v} \right.$$

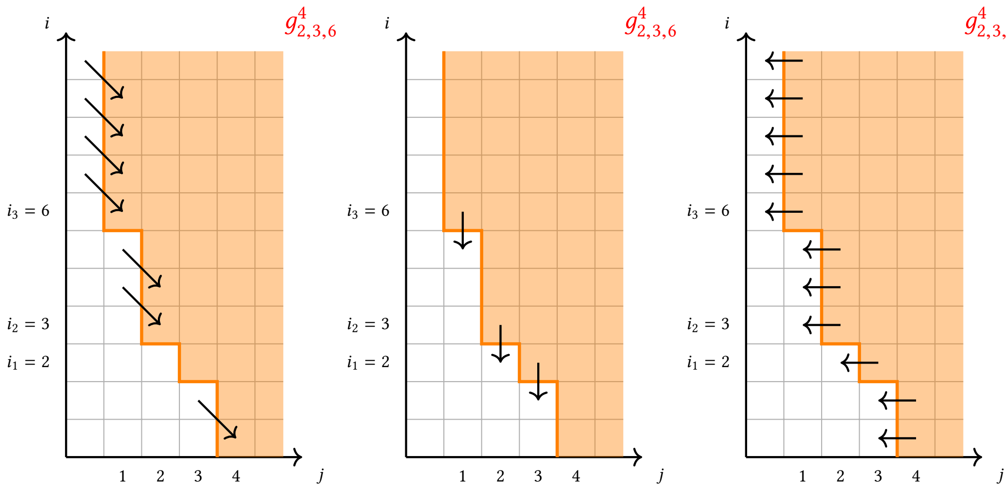

Fig. 2. Illustration of change to $g^4_{2,3,6}(h(t))$ due to phase changes (left), service completions in phase 1 (middle) and service completions in phase 2 (right).

$$+ 1[j \geq s+1] \sum_{k=i_s+1}^{\infty} (g^j_{i_1,\ldots,i_s,k} - g^j_{i_1,\ldots,i_s,k+1}) \frac{k}{k+j-(s+1)} \Bigg).$$

Due to (5), this is the same as

$$1[j \geq 1] p_1 \mu_1 \Bigg(\sum_{v=1}^{s} \sum_{k=i_{v-1}+1}^{i_v - 1} (g^j_{i_1,\ldots,i_{v-1},k,i_{v+1},\ldots,i_s} - g^j_{i_1,\ldots,i_{v-1},k+1,i_{v+1},\ldots,i_s}) \frac{k}{k+j-v}$$
$$+ 1[j \geq s+1] \sum_{k=i_s+1}^{\infty} (g^j_{i_1,\ldots,i_s,k} - g^j_{i_1,\ldots,i_s,k+1}) \frac{k}{k+j-(s+1)} \Bigg). \quad (19)$$

Given $i_1, \ldots, i_s$ (with $i_0 = 0$ and $i_{s+1} = \infty$), we now define a_i as follows:

$$a_i = \frac{i}{i+j-v} - \frac{i-1}{i+j-v-1} \geq 0,$$

for $v = 1, \ldots, s+1$ and $i_{v-1} + 2 \leq i \leq i_v - 1$. Using (8), (19) can be written as

$$1[j \geq 1] p_1 \mu_1 \Bigg(\sum_{v=1}^{s} \Bigg(1[i_v - i_{v-1} \geq 2] g^j_{i_1,\ldots,i_{v-1},i_{v-1}+1,i_{v+1},\ldots,i_s} \frac{i_{v-1}+1}{i_{v-1}+1+j-v}$$
$$+ \sum_{k=i_{v-1}+2}^{i_v - 1} g^j_{i_1,\ldots,i_{v-1},k,i_{v+1},\ldots,i_s} a_k - 1[i_v - i_{v-1} \geq 2] g^j_{i_1,\ldots,i_s} \frac{i_v - 1}{i_v - 1 + j - v} \Bigg)$$
$$+ 1[j \geq s+1] \Bigg(g^j_{i_1,\ldots,i_s,i_s+1} \frac{i_s+1}{i_s+j-s} + \sum_{k=i_s+2}^{\infty} g^j_{i_1,\ldots,i_s,k} a_k - g^j_{i_1,\ldots,i_s} \Bigg) \Bigg), \quad (20)$$

which shows that if $g^j_{i_1,\ldots,i_s}(h(t)) = g^j_{i_1,\ldots,i_s}(\tilde{h}(t))$ with $h(t) \leq_C \tilde{h}(t)$, then the drift of $g^j_{i_1,\ldots,i_s}(\tilde{h}(t))$ due to the phase changes is at least as large as the drift of $g^j_{i_1,\ldots,i_s}(h(t))$ due to the phase changes as only the $g^j_{i_1,\ldots,i_s}$ terms have a negative coefficient in the above expression.

Job completions decrease $g^j_{i_1,\dots,i_s}(h(t))$ in the following two ways as illustrated in Figure 2(middle) and (right):

- when there is a job completion of a job in phase 1 in a server with exactly i_k jobs in phase 1 and exactly $j-k$ jobs in phase 2 for some $k \in \{1,\dots,s\}$;
- when there is a job completion of a job in phase 2 in a server with between i_k and $i_{k+1}-1$ jobs in phase 1 and exactly $j-k$ jobs in phase 2 for some $k \in \{0,\dots,s-1[j=s]\}$.

Set $i_{s+1}=\infty$. The change to $g^j_{i_1,\dots,i_s}(h(t))$ due to service completions can therefore be written as

$$-\nu_1 \sum_{k=1}^{s} w_{i_k,j-k}\frac{i_k}{i_k+j-k} \tag{21}$$

$$-\nu_2 \sum_{k=0}^{s-1[j=s]} \sum_{i=i_k}^{i_{k+1}-1} w_{i,j-k}\frac{j-k}{i+j-k}. \tag{22}$$

Note, that we can drop $-1[j=s]$ from (22) as $j-k=0$ in such case. We now rewrite both these expressions to show that combined they are such that only the $g^j_{i_1,\dots,i_s}$ terms have negative coefficients.

Using Lemma 4.6, we have that (21) is equal to

$$-\nu_1 g^j_{i_1,\dots,i_s}\frac{i_s}{i_s+j-s} + \nu_1\sum_{k=0}^{s-1} g^j_{i_1,\dots,i_k,i_{k+1}+1,\dots,i_s+1}\left(\frac{i_{k+1}}{i_{k+1}+j-k-1}-\frac{i_k}{i_k+j-k}\right). \tag{23}$$

Note that the coefficients appearing in the sum are positive due to (14).

We now proceed with (22). For ease of presentation we assume we have $i_{k+1}-i_k \geq 2$ as the general case is tedious. The full proof can be found in Appendix A. So suppose $i_{k+1}-i_k \geq 2$ for all $k \in \{0,\dots,s-1\}$. We can reorder the terms in (22) as

$$-\nu_2\left(w_{0,j}+\sum_{k=0}^{s}\sum_{i=i_k+1}^{i_{k+1}-2} w_{i,j-k}\frac{j-k}{i+j-k}+\sum_{k=0}^{s-1} w_{i_{k+1}-1,j-k}\frac{j-k}{i_{k+1}-1+j-k}+\sum_{k=1}^{s} w_{i_k,j-k}\frac{j-k}{i_k+j-k}\right),$$

by making use of the fact that $i_{s+1}=\infty$. By means of (7) and (6), we obtain

$$\begin{aligned}-\nu_2\Bigg((g^j-g^{j+1}_1)&+\sum_{k=0}^{s}\sum_{i=i_k+1}^{i_{k+1}-2}(g^{j+1}_{i_1,\dots,i_k,i}-g^{j+1}_{i_1,\dots,i_k,i+1})\frac{j-k}{i+j-k}\\&+\sum_{k=1}^{s}\left(w_{i_k-1,j-k+1}\frac{j-k+1}{i_k+j-k}+w_{i_k,j-k}\frac{j-k}{i_k+j-k}\right)\Bigg).\end{aligned}$$

As

$$w_{i_k-1,j-k+1}+w_{i_k,j-k}=g^{j+1}_{i_1,\dots,i_{k-1},i_k-1,i_k}-g^{j+1}_{i_1,\dots,i_k,i_k+1},$$

we get

$$\begin{aligned}&-\nu_2\Bigg((g^j-g^{j+1}_1)+\sum_{k=0}^{s}\sum_{i=i_k+1}^{i_{k+1}-2}(g^{j+1}_{i_1,\dots,i_k,i}-g^{j+1}_{i_1,\dots,i_k,i+1})\frac{j-k}{i+j-k}\\&+\sum_{k=1}^{s}\left(g^{j+1}_{i_1,\dots,i_{k-1},i_k-1,i_k}-g^{j+1}_{i_1,\dots,i_k,i_k+1}\right)\frac{j-k+1}{i_k+j-k}-\sum_{k=1}^{s} w_{i_k,j-k}\frac{1}{i_k+j-k}\Bigg).\end{aligned}$$

This can be restated using (5) as

$$-\nu_2\Bigg((g^{j}_{i_1,\dots,i_s}-g^{j+1}_{1,i_1,\dots,i_s})+\sum_{k=0}^{s}\sum_{i=i_k+1}^{i_{k+1}-2}(g^{j+1}_{i_1,\dots,i_k,i,i_{k+1},\dots,i_s}-g^{j+1}_{i_1,\dots,i_k,i+1,i_{k+1},\dots,i_s})\frac{j-k}{i+j-k}$$
$$+\sum_{k=1}^{s}(g^{j+1}_{i_1,\dots,i_{k-1},i_k-1,i_k,\dots,i_s}-g^{j+1}_{i_1,\dots,i_k,i_k+1,i_{k+1},\dots,i_s})\frac{j-k+1}{i_k+j-k}-\sum_{k=1}^{s}w_{i_k,j-k}\frac{1}{i_k+j-k}\Bigg).$$

By adding and subtracting two sums we find

$$-\nu_2\Bigg((g^{j}_{i_1,\dots,i_s}-g^{j+1}_{1,i_1,\dots,i_s})$$
$$+\sum_{k=0}^{s}\sum_{i=i_k+1}^{i_{k+1}-1}g^{j+1}_{i_1,\dots,i_k,i,i_{k+1},\dots,i_s}\frac{j-k}{i+j-k}-\sum_{k=0}^{s}\sum_{i=i_k}^{i_{k+1}-2}g^{j+1}_{i_1,\dots,i_k,i+1,i_{k+1},\dots,i_s}\frac{j-k}{i+j-k}$$
$$-\left(\sum_{k=1}^{s}g^{j+1}_{i_1,\dots,i_k,i_k+1,i_{k+1},\dots,i_s}\frac{j-k+1}{i_k+j-k}-\sum_{k=0}^{s}g^{j+1}_{i_1,\dots,i_k,i_k+1,i_{k+1},\dots,i_s}\frac{j-k}{i_k+j-k}\right)$$
$$-\left(\sum_{k=0}^{s}g^{j+1}_{i_1,\dots,i_k,i_{k+1}-1,i_{k+1},\dots,i_s}\frac{j-k}{i_{k+1}-1+j-k}-\sum_{k=1}^{s}g^{j+1}_{i_1,\dots,i_{k-1},i_k-1,i_k,\dots,i_s}\frac{j-k+1}{i_k+j-k}\right)$$
$$-\sum_{k=1}^{s}w_{i_k,j-k}\frac{1}{i_k+j-k}\Bigg).$$

Combining both double sums and keeping in mind that $i_{s+1}=\infty$, the above expression is equivalent to

$$-\nu_2 g^{j}_{i_1,\dots,i_s}+\nu_2\sum_{k=0}^{s}\sum_{i=i_k+1}^{i_{k+1}-1}g^{j+1}_{i_1,\dots,i_k,i,i_{k+1},\dots,i_s}\left(\frac{j-k}{i+j-k-1}-\frac{j-k}{i+j-k}\right)$$
$$+\nu_2\sum_{k=1}^{s}g^{j+1}_{i_1,\dots,i_k,i_k+1,i_{k+1},\dots,i_s}\left(\frac{j-k+1}{i_k+j-k}-\frac{j-k}{i_k+j-k}\right)$$
$$+\nu_2\sum_{k=1}^{s}g^{j+1}_{i_1,\dots,i_{k-1},i_k-1,i_k,\dots,i_s}\left(\frac{j-k+1}{i_k+j-k}-\frac{j-k+1}{i_k+j-k}\right)+\nu_2\sum_{k=1}^{s}w_{i_k,j-k}\frac{1}{i_k+j-k},$$
$$=-\nu_2 g^{j}_{i_1,\dots,i_s}+\nu_2\sum_{k=0}^{s}\sum_{i=i_k+1}^{i_{k+1}-1}g^{j+1}_{i_1,\dots,i_k,i,i_{k+1},\dots,i_s}\left(\frac{j-k}{i+j-k-1}-\frac{j-k}{i+j-k}\right)$$
$$+\nu_2\sum_{k=1}^{s}g^{j+1}_{i_1,\dots,i_k,i_k+1,i_{k+1},\dots,i_s}\frac{1}{i_k+j-k}+\nu_2\sum_{k=1}^{s}w_{i_k,j-k}\frac{1}{i_k+j-k},\tag{24}$$

We still need to deal with the term $\nu_2\sum_{k=1}^{s}w_{i_k,j-k}\frac{1}{i_k+j-k}$. Using Lemma 4.6, we find that this term equals

$$\nu_2 g^{j}_{i_1,\dots,i_s}\frac{1}{i_s+j-s}-\nu_2\sum_{k=0}^{s-1}g^{j}_{i_1,\dots,i_k,i_{k+1}+1,\dots,i_s+1}\Big(\frac{1}{i_{k+1}+j-k-1}-\frac{1}{i_k+j-k}\Big),$$

which has the same form as (23). This shows that the term associated with the service completions from phase 2 is also monotone when $i_{k+1}-i_k\geq 2$ for $k=0,\dots,s$. Note that we did not rely on the fact that $\nu_1 > \nu_2$, however, this requirement is necessary for the full proof in the Appendix where we may have $i_{k+1} = i_k + 1$ for some k values. □

THEOREM 5.3. *Assume Assumptions 1-3 hold and $\nu_1 > \nu_2$, then π is a global attractor of the set of ODEs given by* (1)*, meaning $h(t)$ converges to π as t tends to infinity for any $h_0 \in \Omega_B$ with $h(0) = h_0$.*

PROOF. The proof is similar to [11, Theorem 4]. Define $(h^{(u)})_{i,j} = 1$ and $(h^{(\ell)})_{i,j} = 0$ for $0 < i + j \leq B$, then $h^{(\ell)} \leq_C h \leq_C h^{(u)}$ for all $h \in \Omega_B$. By Proposition 5.2 it suffices to show that $h(t)$ converges to π when $h(0) = h^{(\ell)}$ and when $h(0) = h^{(u)}$ as the trajectories of other initial states h_0 must remain between these two trajectories.

We prove the convergence when $h(0) = h^{(\ell)}$, the proof for $h^{(u)}$ is analogous. First note that $h(0) = h^{(\ell)} \leq_C h(t-s)$ holds for $0 < s < t$, as $h^{(\ell)} \leq_C h$ for all $h \in \Omega_B$. Hence, by Proposition 5.2 $h(s) \leq_C h(t)$ for $0 < s < t$, as $h(t)$ is the state at time s if we start in state $h(t-s)$. The theory of monotone dynamical systems (see [10, Theorem 1.4]) now implies that $h(t)$ converges to a fixed point as Ω_B is a compact set. Due to Assumption 3, $h(t)$ must converge to π when $h(0) = h^{(\ell)}$. □

6 THE SUPERMARKET MODEL

In this section we show that Assumptions 1-3 hold for the supermarket model with processor sharing. We start by describing the expressions for $f_{i,j}(h(t))$. We first note that $f_{0,j}(h(t)) = 0$ as new jobs start service in phase 1 (that is, $\alpha = (1, 0)$ when using the Coxian representation). So suppose $i \geq 1$. The probability that all d chosen servers have at least $i + j - 1$ jobs but not all have at least $i + j$ jobs is $h^d_{i+j-1,0}(t) - h^d_{i+j,0}(t)$. In other words, this is the probability that at least one chosen server has exactly $i + j - 1$ jobs. As

$$\frac{h_{i-1,j}(t) - h_{i,j}(t)}{h_{i+j-1,0}(t) - h_{i+j,0}(t)}$$

is the probability that a server with exactly $i + j - 1$ jobs has at least j jobs in phase 2, the arrival terms are given by

$$\begin{aligned} f_{i,j}(h(t)) &= \lambda \left(h^d_{i+j-1,0}(t) - h^d_{i+j,0}(t)\right) \frac{h_{i-1,j}(t) - h_{i,j}(t)}{h_{i+j-1,0}(t) - h_{i+j,0}(t)} \\ &= \lambda \left(\sum_{\ell=0}^{d-1} h^\ell_{i+j-1,0}(t) h^{d-1-\ell}_{i+j,0}(t)\right)(h_{i-1,j}(t) - h_{i,j}(t)), \end{aligned} \tag{25}$$

where we have used the identity $(a^d - b^d)/(a-b) = \sum_{\ell=0}^{d-1} a^\ell b^{d-1-\ell}$. Define $\Psi_{i,j}(h(t)) = \frac{d}{dt} h_{i,j}(t)$ and $\Psi(h) = \left[\Psi_{i,j}(h)\right]_{i,j=0}^{\infty}$. We first show that Ψ is Lipschitz continuous on Ω_B. We use the supremum metric on Ω_B:

$$\mathbf{d}(h, \tilde{h}) = \sup_{i,j=0}^{\infty} |h_{i,j} - \tilde{h}_{i,j}|. \tag{26}$$

PROPOSITION 6.1. *The drift Ψ is Lipschitz continuous on Ω_B, meaning Assumption 1 is met when $f_{i,j}(h(t))$ is defined as in* (25)*.*

PROOF. Let $h, \tilde{h} \in \Omega_B$ and let $w, \tilde{w}$ be the corresponding vectors as defined in Section 3. As $w_{i,j} = (h_{i,j} - h_{i-1,j+1}) - (h_{i+1,j} - h_{i,j+1})$ for $i \geq 1$ and $w_{0,j} = h_{0,j} - h_{1,j}$, we have

$$\sup_{i,j=0}^{\infty} |w_{i,j} - \tilde{w}_{i,j}| \leq 4\mathbf{d}(h, \tilde{h}). \tag{27}$$

We have

$$\nu_2 \sup_{i,j=0}^{\infty} \left| \sum_{k=i+1}^{\infty} (w_{k,j} - \tilde{w}_{k,j}) \frac{j}{k+j} \right| \leq \nu_2 \sup_{i,j=0}^{\infty} \sum_{k=i+1}^{\infty} \left| w_{k,j} - \tilde{w}_{k,j} \right|$$
$$\leq \nu_2 (B+1) \sup_{i,j=0}^{\infty} \left| w_{i,j} - \tilde{w}_{i,j} \right| \leq 4\nu_2 (B+1) \mathbf{d}(h, \tilde{h}),$$

where we have used (27) in the last inequality. Proceeding similarly we get

$$\mathbf{d}(\Psi(h), \Psi(\tilde{h})) \leq 4(B+1)(\mu_1 + 2\mu_2)\mathbf{d}(h, \tilde{h})$$
$$+ \lambda \sup_{i,j=0}^{\infty} \left| (h_{i,j} - h_{i+1,j}) \sum_{\ell=0}^{d-1} h_{i+j,0}^{\ell} h_{i+j+1,0}^{d-1-\ell} - (\tilde{h}_{i,j} - \tilde{h}_{i+1,j}) \sum_{\ell=0}^{d-1} \tilde{h}_{i+j,0}^{\ell} \tilde{h}_{i+j+1,0}^{d-1-\ell} \right|.$$

We now use the inequality $|a_1^{m_1} a_2^{m_2} - b_1^{m_1} b_2^{m_2}| \leq m_1 |a_1 - b_1| + m_2 |a_2 - b_2|$, for $0 \leq a_1, a_2, b_1, b_2 \leq 1$ and $m_1, m_2 \in \mathbb{N} \setminus \{0\}$, to find that

$$\sup_{i,j=0}^{\infty} \left| (h_{i,j} - h_{i+1,j}) \sum_{\ell=0}^{d-1} h_{i+j,0}^{\ell} h_{i+j+1,0}^{d-1-\ell} - (\tilde{h}_{i,j} - \tilde{h}_{i+1,j}) \sum_{\ell=0}^{d-1} \tilde{h}_{i+j,0}^{\ell} \tilde{h}_{i+j+1,0}^{d-1-\ell} \right|$$
$$\leq \sup_{i,j=0}^{\infty} |h_{i,j} - h_{i+1,j} - (\tilde{h}_{i,j} - \tilde{h}_{i+1,j})| + \sup_{i,j=0}^{\infty} \left| \sum_{\ell=0}^{d-1} (h_{i+j,0}^{\ell} h_{i+j+1,0}^{d-1-\ell} - \tilde{h}_{i+j,0}^{\ell} \tilde{h}_{i+j+1,0}^{d-1-\ell}) \right|$$
$$\leq 2\mathbf{d}(h, \tilde{h}) + \sup_{i,j=0}^{\infty} \sum_{\ell=0}^{d-1} \left| h_{i+j,0}^{\ell} h_{i+j+1,0}^{d-1-\ell} - \tilde{h}_{i+j,0}^{\ell} \tilde{h}_{i+j+1,0}^{d-1-\ell} \right|.$$

Using the above mentioned inequality once more, we get

$$\sup_{i,j=0}^{\infty} \sum_{\ell=0}^{d-1} \left| h_{i+j,0}^{\ell} h_{i+j+1,0}^{d-1-\ell} - \tilde{h}_{i+j,0}^{\ell} \tilde{h}_{i+j+1,0}^{d-1-\ell} \right|$$
$$\leq \sup_{i,j=0}^{\infty} \sum_{\ell=0}^{d-1} \Big(\ell |h_{i+j,0} - \tilde{h}_{i+j,0}| + (d-1-\ell) |h_{i+j+1,0} - \tilde{h}_{i+j+1,0}| \Big)$$
$$\leq 2d^2 \mathbf{d}(h, \tilde{h}).$$

To conclude, we have

$$\mathbf{d}(\Psi(h), \Psi(\tilde{h})) \leq \big(4(B+1)(\mu_1 + 2\mu_2) + 2\lambda(d^2 + 1) \big) \, \mathbf{d}(h, \tilde{h}).$$

□

We now proceed with Assumption 2. Let $j, s, i_1, \ldots, i_s$ be as in Definition 4.2. Define

$$b_k = \lambda \sum_{\ell=0}^{d-1} h_{i_k+j-k-1,0}^{\ell}(t) h_{i_k+j-k,0}^{d-1-\ell}(t),$$

to simplify the notation. Note, that $b_k > b_{k+1}$ if $i_{k+1} - i_k \geq 2$ and $b_k = b_{k+1}$ if $i_{k+1} - i_k = 1$. We then have that $F_{i_1,\ldots,i_s}^{j}(h(t))$, as defined in Definition 5.1, is given by

$$F_{i_1,\ldots,i_s}^{j}(h(t)) = f_{0,j}(h(t)) + \sum_{k=1}^{s} (f_{i_k,j-k}(h(t)) - f_{i_k-1,j-k+1}(h(t)))$$
$$= \lambda \sum_{k=1}^{s} \Big(h_{i_k+j-k-1,0}^{d}(t) - h_{i_k+j-k,0}^{d}(t) \Big)$$

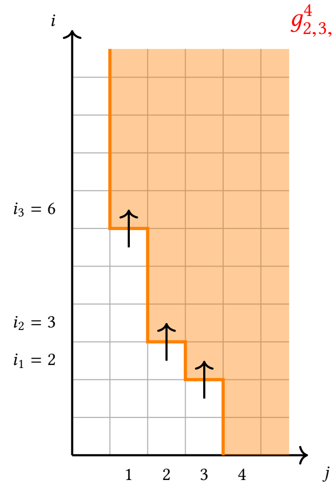

Fig. 3. Illustration of change to $g^4_{2,3,6}(h(t))$ due to arrivals with JSQ(d).

$$\cdot \frac{(h_{i_k-1,j-k}(t) - h_{i_k,j-k}(t)) - 1[i_k \geq 2](h_{i_k-2,j-k+1}(t) - h_{i_k-1,j-k+1}(t))}{h_{i_k+j-k-1,0}(t) - h_{i_k+j-k,0}(t)}$$
$$= \sum_{k=1}^{s} b_k w_{i_k-1,j-k}(h(t)).$$

The change due to arrivals is illustrated in Figure 3.

Proposition 6.2. *Assumption 2 holds for the system of ODEs* (1) *with* $f_{i,j}(h(t))$ *specified by* (25).

Proof. It suffices to show that $F^j_{i_1,\dots,i_s}(h(t))$ is non-decreasing in $g^{j'}_{i'_1,\dots,i'_{s'}}(h(t))$ for all sets of indices $\{j', i'_1, \dots, i'_{s'}\}$, as in (4), different from $\{j, i_1, \dots, i_s\}$. Suppose first that $i_1 \geq 2$. For ease of notation set $b_0 = b_{s+1} = 0$. By using Lemma 4.6, we find

$$\begin{aligned} F^j_{i_1,\dots,i_s}(h(t)) &= \sum_{k=1}^{s} b_k w_{i_k-1,j-k}(h(t)) \\ &= g^j_{i_1-1,\dots,i_s-1}(h(t))b_s - \sum_{k=0}^{s-1} g^j_{i_1-1,\dots,i_k-1,i_{k+1},\dots,i_s}(h(t))\Big(b_{k+1} - b_k\Big) \\ &= -g^j_{i_1,\dots,i_s}(h(t))b_1 + \sum_{k=1}^{s} g^j_{i_1-1,\dots,i_k-1,i_{k+1},\dots,i_s}(h(t))\Big(b_k - b_{k+1}\Big). \end{aligned}$$

Suppose now $i_1 = 1$. We have

$$F^j_{i_1,\dots,i_s}(h(t)) = b_1 w_{0,j-1}(h(t)) + \sum_{k=2}^{s} b_k w_{i_k-1,j-k}(h(t)).$$

Using (7) and (6), this is equal to

$$b_1(g^{j-1}(h(t)) - g^j_1(h(t))) + \sum_{k=2}^{s} b_k(g^{j-1}_{i_2-1,\dots,i_k-1}(h(t)) - g^{j-1}_{i_2-1,\dots,i_{k-1}-1,i_k}(h(t))).$$

By (5), we can write this as

$$b_1(g^{j-1}_{i_2,\ldots,i_s}(h(t)) - g^{j}_{i_1,\ldots,i_s}(h(t))) + \sum_{k=2}^{s} b_k(g^{j-1}_{i_2-1,\ldots,i_k-1,i_{k+1},\ldots,i_s}(h(t)) - g^{j-1}_{i_2-1,\ldots,i_{k-1}-1,i_k,\ldots,i_s}(h(t)))$$

$$= -b_1 g^{j}_{i_1,\ldots,i_s}(h(t)) + \sum_{k=1}^{s} g^{j-1}_{i_2-1,\ldots,i_k-1,i_{k+1},\ldots,i_s}(h(t))\,(b_k - b_{k+1})\,.$$

This finishes the proof. □

We now present a fixed point π of the system of the ODEs given in (1) when $f_{i,j}(h(t))$ is given by (25) and then prove that it is the unique fixed point. As $h_{i,j}(t) = \sum_{k\geq i+j}\sum_{\ell=j}^{k} w_{k-\ell,\ell}(h(t))$, finding a fixed point of (1) is equivalent to finding a fixed point π^w of the corresponding set of equations for $\frac{d}{dt}w_{i,j}(h(t))$. We have

$$\begin{aligned}
\frac{d}{dt}w_{i,j}(h(t)) = {} & 1[i\geq 1]\lambda\left(h^d_{i+j-1,0}(t) - h^d_{i+j,0}(t)\right)\frac{w_{i-1,j}(h(t))}{h_{i+j-1,0}(t) - h_{i+j,0}(t)} \\
& - \lambda\left(h^d_{i+j,0}(t) - h^d_{i+j+1,0}(t)\right)\frac{w_{i,j}(h(t))}{h_{i+j,0}(t) - h_{i+j+1,0}(t)} \\
& + 1[j\geq 1]p_1\mu_1 w_{i+1,j-1}(h(t))\frac{i+1}{i+j} - \mu_1 w_{i,j}(h(t))\frac{i}{i+j} - \mu_2 w_{i,j}(h(t))\frac{j}{i+j} \\
& + \mu_2 w_{i,j+1}(h(t))\frac{j+1}{i+j+1} + (1-p_1)\mu_1 w_{i+1,j}(h(t))\frac{i+1}{i+j+1},
\end{aligned} \tag{28}$$

for $i,j\geq 0$ and $i+j\leq B$, where $w_{i,j}(h(t)) = 0$ for $i+j > B$. Consider the set of ODEs for $i = 1,\ldots,B$ given by

$$\frac{d}{dt}\hat{h}_i(t) = \lambda(\hat{h}_{i-1}(t)^d - \hat{h}_i(t)^d) - (\hat{h}_i(t) - \hat{h}_{i+1}(t)) \tag{29}$$

and set $\hat{h}_0(t) = 1$ and $\hat{h}_i(t) = 0$, for $i > B$. Notice this set of ODEs corresponds to the mean field limit of the supermarket model with FCFS or processor sharing service, exponential job sizes and a finite buffer of size B [14].

Proposition 6.3. *The set of ODEs given by* (29) *has a unique fixed point $\hat{\pi}$. Further $\hat{\pi}_i \in (0, \lambda^{(d^i-1)/(d-1)})$ and $\hat{\pi}_i > \hat{\pi}_{i+1}$.*

Proof. Summing the equations in (29) from $i = k$ to B yields that any fixed point $\hat{\pi}$ satisfies

$$\hat{\pi}_k = \lambda(\hat{\pi}^d_{k-1} - \hat{\pi}^d_B),$$

for $k = 2,\ldots,B$ and $\hat{\pi}_1 = \lambda(1-\hat{\pi}^d_B)$. Define $H_1(x) = \lambda(1-x^d)$ and $H_k(x) = \lambda(H_{k-1}(x)^d - x^d)$ for $k = 2,\ldots,B$, then $\hat{\pi}_k = H_k(\hat{\pi}_B)$. Using induction on k one now readily shows that $H_k(0) = \lambda^{1+d+\ldots+d^{k-1}} = \lambda^{(d^k-1)/(d-1)} > 0$.

We first consider the case where d is odd. As $H_{k-1}(x)^{d-1} \geq 0$ for d odd and $H_k'(x) = \lambda d H_{k-1}(x)^{d-1}H_{k-1}'(x) - \lambda d x^{d-1}$, we immediately have by induction on k that $H_k(x)$ is decreasing on $[0,1]$ with $H_k(1)\leq 0$. As $\hat{\pi}_B = H_B(\hat{\pi}_B)$, this implies that there exists a unique solution for $\hat{\pi}_B \in (0, \lambda^{(d^B-1)/(d-1)})$ and therefore at most one fixed point as $\hat{\pi}_k = H_k(\hat{\pi}_B)$. Further, we see that $\hat{\pi}_k \leq \lambda^{(d^k-1)/(d-1)}$ as $H_k(0) = \lambda^{(d^k-1)/(d-1)}$ and $H_k(x)$ is decreasing. Finally, $\hat{\pi}_k < \hat{\pi}_{k-1}$ as $\hat{\pi}_k = \lambda(\hat{\pi}^d_{k-1} - \hat{\pi}^d_B)$ with $\hat{\pi}_B$ positive.

Now assume d is even. $H_1(x)$ is clearly decreasing and positive on $[0,1]$, therefore $H_2(x)$ is decreasing on $[0,1]$ and $H_2(1) = -\lambda < 0$. Let ξ_2 be the unique root of $H_2(x)$ in $(0,1)$. For $x\in(\xi_2,1]$ we have $-x < H_2(x) < 0$. We now find using induction on k that for $k = 3,\ldots,B$

(1) $H_k(x)$ is decreasing on $[0, \xi_{k-1})$ as $H_k'(x) = \lambda d H_{k-1}(x)^{d-1} H_{k-1}'(x) - \lambda d x^{d-1}$ and $H_{k-1}(x)$ is positive and decreasing on $[0, \xi_{k-1})$,
(2) $H_k(\xi_{k-1}) = -\lambda \xi_{k-1}^d < 0$ and $-x < -\lambda x^d < H_k(x) < 0$ for $x \in [\xi_{k-1}, 1]$, where we use (i) that $H_{k-1}(x)^d \geq 0$ for d even to find that $-\lambda x^d < H_k(x)$ and (ii) that $-x < H_{k-1}(x)$ yields $x^d > H_{k-1}(x)^d$ for d even, that is, $H_k(x) < 0$.
(3) $H_k(x)$ has a unique root ξ_k on $[0, 1]$ with $\xi_k < \xi_{k-1}$.

The proof now proceeds as in the case with d odd. □

We now define π^w as

$$\pi_{i,j}^w = (\hat{\pi}_{i+j} - \hat{\pi}_{i+j+1}) \binom{i+j}{i} \left(\frac{1}{\mu_1}\right)^i \left(\frac{p_1}{\mu_2}\right)^j, \tag{30}$$

and show that it is a fixed point of (28). Note that π^w depends on λ via $\hat{\pi}$. If we then define $\pi_{i,j} = \sum_{k \geq i+j} \sum_{\ell=j}^k \pi_{k-\ell,\ell}^w$, then π is a fixed point of (1) and

$$\pi_{i,0} = \sum_{k \geq i} \sum_{\ell=0}^{k} \pi_{k-\ell,\ell}^w = \sum_{k \geq i} (\hat{\pi}_k - \hat{\pi}_{k+1}) \left(\frac{1}{\mu_1} + \frac{p_1}{\mu_2}\right)^k = \hat{\pi}_i,$$

as the mean job size equals one, that is, $1/\mu_1 + p_1/\mu_2 = 1$. Hence the probability of having i or more jobs in the fixed point π is the same as in the exponential case.

Proposition 6.4. *π^w is a fixed point of* (28).

Proof. The proof is presented in Appendix B. □

Proposition 6.5. *π^w is the unique fixed point of* (28) *and therefore Assumption 3 holds for the set of ODEs in* (1) *with $f_{i,j}(h(t))$ specified by* (25).

Proof. The proof proceeds similar to [18, Theorem 3]. We first argue that any fixed point θ must have the same form as in (30), that is, it can be written as

$$\theta_{i,j} = (\hat{\theta}_{i+j} - \hat{\theta}_{i+j+1}) \binom{i+j}{i} \left(\frac{1}{\mu_1}\right)^i \left(\frac{p_1}{\mu_2}\right)^j. \tag{31}$$

Let $\lambda_{i+j} = \lambda(\theta_{i+j,0}^d - \theta_{i+j+1,0}^d)/(\theta_{i+j,0} - \theta_{i+j+1,0})$ and replace $\lambda \left(h_{i+j-1,0}^d(t) - h_{i+j,0}^d(t)\right) / (h_{i+j-1,0}(t) - h_{i+j,0}(t))$ in the set of ODEs given by (28) by λ_{i+j-1} and $\lambda \left(h_{i+j,0}^d(t) - h_{i+j+1,0}^d(t)\right) / (h_{i+j,0}(t) - h_{i+j+1,0}(t))$ by λ_{i+j}. Then, θ is also a fixed point of this new set of ODEs. However this system of ODEs corresponds an M/PH/1 queue with a pre-specified arrival rate that depends on the queue length and processor sharing service. As such a queue is insensitive to the job size distribution [3], it has a unique fixed point of the form given in (31). It now suffices to argue that $\hat{\theta}_j = \hat{\pi}_j$, where $\hat{\pi}$ is the unique solution of (29).

As θ has the form given in (31), we can repeat the proof of the previous theorem to show that (41) holds with $\hat{\pi}$ replaced by $\hat{\theta}$ for any i and j and therefore also for $i = 0$, which means $\hat{\theta}$ is a fixed point of (29). The proof is then completed due to Proposition 6.3. □

7 ASYMPTOTIC INSENSITIVITY

We now present our main result, having established global attraction of the fixed point π, the proof is quite standard. Let $\pi^{(N)}$ be the stationary measure associated to the Markov chain $X^{(N)}(t)$ that captures the fraction of the servers with $i + j$ or more jobs, where at least j of these jobs are in phase 2 in a system with N servers and initial state $X^{(N)}(0) \in \Omega_B$.

Theorem 7.1. *The limiting queue length distribution of the supermarket model with processor sharing service is insensitive to the job size distribution within the class of hyperexponential distributions of order 2, that is, the sequence $\pi^{(N)}$ converges weakly to the Dirac measure associated with the fixed point π. In other words, the limiting queue length distribution is given by the unique fixed point $\hat{\pi}$ of* (29).

Proof. By the Lipschitz continuity and Kurtz' theorem [5, Chapter 11] the sample paths $X^{(N)}(t)$ of the stochastic system consisting of N servers converge in probability to the unique solution of the set of ODEs given by (1) with $f_{i,j}(h(t))$ specified by (25) over any finite time scale $(0,T]$, that is,

$$\lim_{N\to\infty} \sup_{t\leq T} ||X^N(t) - h(t)|| = 0,$$

in probability if $\lim_{N\to\infty} X^N(0) = h_0$, where $h(0) = h_0$.

As Ω_B is compact the sequence of stationary measures $\pi^{(N)}$ is tight. Hence, Prokhorov's theorem implies that any subsequence of the sequence $\pi^{(N)}$ has a further subsequence that converges to some measure on Ω_B. Now [7, Theorem 4], implies that *any* limit point of the these further subsequences of $\pi^{(N)}$ has support on the compact closure of the set of accumulation points of the set of ODEs for all initial conditions $h_0 \in \Omega_B$. By Theorem 5.3 the only accumulation point is the fixed point π, which proves that all these further subsequences converge to the same limit point, being the Dirac measure δ_π of the fixed point π. This implies that the sequence of measures $\pi^{(N)}$ converges weakly to the Dirac measure δ_π. □

An interesting question at this stage is whether this result can be generalized easily to hyperexponential distributions of order $r > 2$. We now argue that this does not appear to be the case. Assume we have 3 phases, then the state would be captured by the variables $h_{i,j,k}(t)$ that represent the fraction of the servers with at least $i + j + k$ jobs, of which at least $j + k$ are in phase 2 or 3 and at least k are in phase 3 at time t. Similarly, define $w_{i,j,k}(t)$ as the fraction of the servers with exactly i jobs in phase 1, j in phase 2 and k in phase 3. Now consider the state $h(0)$ where $w_{0,1,0}(t) = 1$, meaning all servers contain 1 job and this job is in phase 2. Further, consider the state $\tilde{h}(t)$ with $\tilde{w}_{99,1,0}(t) = 1$, meaning all servers contain 100 jobs, 99 in phase 1 and 1 in phase 2. If we generalize the partial order $\leq_C$ presented in this paper in the obvious manner, then clearly $h(0) \leq_C \tilde{h}(0)$. However, $h(\epsilon) \leq_C \tilde{h}(\epsilon)$ may not hold for ϵ small, meaning the system does not appear to be monotone. To understand this, note that from state $h(0)$ servers are created that contain at least 1 job in phase 3 at a rate $\mu_2 p_2$ as the full server capacity is devoted to a single job in phase 2 in state $h(0)$, while from state $\tilde{h}(0)$ jobs in phase 3 are only created at a rate $\mu_2 p_2/100$, thus $h_{0,0,1}(\epsilon)$ will exceed $\tilde{h}_{0,0,1}(\epsilon)$ for ϵ small enough.

8 TRADITIONAL PUSH

In this section we illustrate that Theorem 5.3 can also be used to prove asymptotic insensitivity of other systems with PS servers. More specifically we focus on the traditional push strategy studied for FCFS servers in [4] and [13, Section VI.A]. We will argue that Assumptions 1 to 3 hold for this strategy in case of PS servers, which allows us to establish asymptotic insensitivity within the class of order-2 hyperexponential distributions by using the same arguments as in the proof of Theorem 7.1.

We consider a system consisting of N servers, each subject to its own local Poisson arrival process with rate $\lambda < 1$. When a job arrives in server n and server n is busy, a single random server n' is probed and the incoming job is immediately transferred to server n' provided that it is idle. Otherwise the job is executed on server n. Note that although the servers are PS servers, a job is fully executed on a single server under this strategy. This is in contrast to the traditional pull or the rate-based strategies in [13], where transferred jobs are always partially executed on one server before being transferred to another PS server. In fact, such partial executions imply that asymptotic insensitive is lost despite the PS service discipline.

We first define the drift terms $f_{i,j}(h)$ corresponding to job arrivals and transfers for the traditional push strategy. As jobs always start service in phase 1, we have $f_{0,j}(h) = 0$ as in the supermarket model. When $i > 0$ and $i + j > 1$, then $h_{i-1,j} - h_{i,j}$ is the fraction of the queues with an exact queue length of $i + j - 1$ with at least j jobs in phase 2. Arrivals in such a queue increase the queue length to $i + j$ provided that the probed server is busy, which occurs with probability $h_{1,0}$. Hence, $\lambda(h_{i-1,j} - h_{i,j})h_{1,0}$ is the rate at which $h_{i,j}$ increases due to arrivals that are not transferred (for $i > 0$ and $i + j > 1$). When $i = 1$ and $j = 0$, $h_{i,j} = h_{1,0}$ is the fraction of busy servers and this fraction increases at rate $\lambda(1 - h_{1,0})$, due to local arrivals in idle servers, plus $\lambda h_{1,0}(1 - h_{1,0})$, due to arrivals in busy servers that are immediately transferred to an idle server. Hence, we have

$$\begin{aligned} f_{i,j}(h) &= 1[i > 0, i + j > 1]\lambda(h_{i-1,j} - h_{i,j})h_{1,0} \\ &\quad + 1[i = 1, j = 0]\lambda\left((1 - h_{1,0}) + h_{1,0}(1 - h_{1,0})\right) \\ &= 1[i > 0]\lambda(h_{i-1,j} - h_{i,j})h_{1,0} + 1[i = 1, j = 0]\lambda(1 - h_{1,0}). \end{aligned} \tag{32}$$

In particular, we have

$$f_{1,0}(h) = \lambda(1 - (h_{1,0})^2). \tag{33}$$

Define $\Xi_{i,j}(h(t)) = \frac{d}{dt}h_{i,j}(t)$ and $\Xi(h) = \left[\Xi_{i,j}(h)\right]_{i,j=0}^{\infty}$. The next two results show that Assumptions 1 and 2 hold.

Proposition 8.1. *The drift Ξ is Lipschitz continuous on Ω_B, meaning Assumption 1 is met when $f_{i,j}(h)$ is defined as in* (32).

Proof. Let $h, \tilde{h} \in \Omega_B$. Proceeding similarly to Proposition 6.1, we get

$$\mathbf{d}(\Xi(h), \Xi(\tilde{h})) \le 4(B + 1)(\mu_1 + 2\mu_2)\mathbf{d}(h, \tilde{h}) + \sup_{i,j=0}^{\infty} \left| f_{i,j}(h) - f_{i,j}(\tilde{h}) \right|.$$

We have

$$\sup_{i,j=0}^{\infty} \left| f_{i,j}(h) - f_{i,j}(\tilde{h}) \right| \le \lambda\mathbf{d}(h, \tilde{h}) + \lambda \sup_{i,j=0}^{\infty} \left| (h_{i-1,j} - h_{i,j})h_{1,0} - (\tilde{h}_{i-1,j} - \tilde{h}_{i,j})\tilde{h}_{1,0} \right|. \tag{34}$$

We now use the inequality $|a_1a_2 - b_1b_2| \le |a_1 - b_1| + |a_2 - b_2|$, for $0 \le a_1, a_2, b_1, b_2 \le 1$ on (34), to find that

$$\sup_{i,j=0}^{\infty} \left| (h_{i-1,j} - h_{i,j})h_{1,0} - (\tilde{h}_{i-1,j} - \tilde{h}_{i,j})\tilde{h}_{1,0} \right| \le \left| h_{1,0} - \tilde{h}_{1,0} \right| + 2\mathbf{d}(h, \tilde{h}) \le 3\mathbf{d}(h, \tilde{h}).$$

To conclude, we have

$$\mathbf{d}(\Xi(h), \Xi(\tilde{h})) \le (4(B + 1)(\mu_1 + 2\mu_2) + 4\lambda)\, \mathbf{d}(h, \tilde{h}).$$

□

Proposition 8.2. *Assumption 2 holds for the system of ODEs* (1) *with $f_{i,j}(h)$ specified by* (32).

Proof. The proof is presented in Appendix C. □

We now proceed by arguing that we have a unique fixed point in Ω_B. As $h_{i,j} = \sum_{k \ge i+j} \sum_{\ell=j}^{k} w_{k-\ell,\ell}$, finding a fixed point of (1) is equivalent to finding a fixed point π^w of the corresponding set of equations for $\frac{d}{dt} w_{i,j}(h(t))$. As there is no ambiguity here, we denote $w(h(t))$ simply as $w(t)$. For the traditional push we have, similar to (28),

$$\begin{aligned}\frac{d}{dt} w_{i,j}(t) = 1[i \geq 1]\lambda(w_{i-1,j}(t) - w_{i,j}(t))(1 - w_{0,0}(t)) + 1[i = 1, j = 0]\lambda w_{0,0}(t) \\ - 1[i = j = 0]\lambda w_{0,0}(t) - 1[i = 0]\lambda w_{0,j}(t)(1 - w_{0,0}(t)) + 1[j \geq 1] p_1 \mu_1 w_{i+1,j-1}(t)\frac{i+1}{i+j} \\ - \mu_1 w_{i,j}(t)\frac{i}{i+j} - \mu_2 w_{i,j}(t)\frac{j}{i+j} + \mu_2 w_{i,j+1}(t)\frac{j+1}{i+j+1} + (1-p_1)\mu_1 w_{i+1,j}(t)\frac{i+1}{i+j+1},\end{aligned} \tag{35}$$

for $i, j \geq 0$ and $i + j \leq B$, where $w_{i,j}(t) = 0$ for $i + j > B$.

We first consider the set of ODEs for the traditional push strategy in case of exponential job sizes (which has the same form as in [13, Section VI.A] for FCFS servers). Let $k_i(t)$ be the fraction of servers with i or more jobs at time t, then

$$\frac{d}{dt} k_i(t) = \lambda(k_{i-1}(t) - k_i(t))k_1(t) - (k_i(t) - k_{i+1}(t)) + 1[i = 1]\lambda(1 - k_1(t)), \tag{36}$$

for $i = 1, \ldots, B$ and set $k_0(t) = 1$ and $k_i(t) = 0$, for $i > B$.

Proposition 8.3. *The set of ODEs given by* (36) *has a unique fixed point $\hat{\pi}$. Further $\hat{\pi}_i \in (0, \lambda^{2i-1})$ and $\hat{\pi}_i > \hat{\pi}_{i+1}$.*

Proof. Summing the equations in (36) from $i = k$ to B yields that any fixed point $\hat{\pi}$ satisfies

$$\hat{\pi}_k = \frac{\lambda^2}{1 + \lambda \hat{\pi}_B}(\hat{\pi}_{k-1} - \hat{\pi}_B),$$

for $k = 2, \ldots, B$ and $\hat{\pi}_1 = \lambda/(1 + \lambda \hat{\pi}_B)$. Define $V_1(x) = \lambda/(1 + \lambda x)$ and $V_k(x) = \frac{\lambda^2}{1+\lambda x}(V_{k-1}(x) - x)$ for $k = 2, \ldots, B$, then $\hat{\pi}_k = V_k(\hat{\pi}_B)$. It is easy to see that $V_k(0) = \lambda^{2k-1} > 0$. By noting that $(V_{i+1}(x) - V_i(x)) = \frac{\lambda^2}{1+\lambda x}(V_i(x) - V_{i-1}(x))$, we find,

$$\begin{aligned}V_k(x) &= \sum_{i=1}^{k-1}(V_{i+1}(x) - V_i(x)) + V_1(x) = \sum_{i=1}^{k-1}\left(\frac{\lambda^2}{1+\lambda x}\right)^{i-1}(V_2(x) - V_1(x)) + V_1(x) \\ &= -\underbrace{\frac{1 - \left(\frac{\lambda^2}{1+\lambda x}\right)^{k-1}}{1 - \frac{\lambda^2}{1+\lambda x}}}_{>0}\underbrace{\frac{\lambda(\lambda^2 x^2 - \lambda^2 + 2\lambda x + 1)}{(1+\lambda x)^2}}_{>0} + \frac{\lambda}{1+\lambda x},\end{aligned}$$

for $x \in [0, 1]$. Hence, $V_k(x) < V_{k-1}(x)$ for $k = 2, \ldots, B$ and $x \in [0, 1]$. One easily checks that $V_2'(x) < 0$ on $[0, 1]$ and $V_2(1) = -\lambda^2/(1 + \lambda)^2 < 0$, meaning $V_2(x)$ has a unique root ξ_2 on $[0, 1]$. We now complete the proof by showing by induction that $V_k(x)$ is decreasing on $[0, \xi_{k-1}]$ and $V_k(x) < 0$ on $[\xi_{k-1}, 1]$, which implies that $V_k(x)$ has a unique root $\xi_k < \xi_{k-1}$ on $[0, 1]$ (as $V_k(0) = \lambda^{2k-1}$) and $V_k(x)$ is negative on $(\xi_k, 1]$.

By definition of $V_k(x)$, we have

$$V_k'(x) = \frac{\lambda^2}{1+\lambda x}V_{k-1}'(x) - \frac{\lambda^3}{(1+\lambda x)^2}V_{k-1}(x) - \frac{\lambda^2}{(1+\lambda x)^2}.$$

Therefore, $V_k'(x)$ is negative if $V_{k-1}'(x) \leq 0$ and $V_{k-1}(x) \geq 0$. By induction this is the case on $[0, \xi_{k-1}]$, meaning $V_k(x)$ is decreasing on $[0, \xi_{k-1}]$ and negative on $[\xi_{k-1}, 1]$ as $V_k(x) < V_{k-1}(x)$. □

Proposition 8.4. *Let $\hat{\pi}$ be the unique fixed point of* (36) *and define π^w as in* (30)*, then π^w is a fixed point of* (35)*.*

Proof. The proof is nearly identical to the proof of Proposition 6.4, that is, by replacing $w_{i,j}(t)$ by $\pi^w_{i,j}$ in (35) with the left hand side set equal to zero, one obtains

$$0 = \left(\lambda(\hat{\pi}_{i+j-1} - \hat{\pi}_{i+j})\hat{\pi}_1 - (\hat{\pi}_{i+j} - \hat{\pi}_{i+j+1}) + 1[i+j=1]\lambda(1-\hat{\pi}_1)\right)\frac{\mu_1 i}{i+j}$$
$$- \left(\lambda(\hat{\pi}_{i+j} - \hat{\pi}_{i+j+1})\hat{\pi}_1 - (\hat{\pi}_{i+j+1} - \hat{\pi}_{i+j+2}) + 1[i+j=0]\lambda(1-\hat{\pi}_1)\right). \tag{37}$$

for $i \geq 0$, which holds as $\hat{\pi}$ is the unique fixed point of (36). □

Proposition 8.5. *π^w is the unique fixed point of* (35) *and therefore Assumption 3 holds for the set of ODEs in* (1) *with $f_{i,j}(h)$ specified by* (32)*.*

Proof. We can repeat the same arguments as in the proof of Proposition 6.5, except that we define $\lambda_{i+j} = \lambda(1-\theta_{0,0})$ for $i+j>0$, $\lambda_0 = \lambda + \lambda(1-\theta_{0,0})$ and rely on Proposition 8.3. Note that the M/PH/1 queue with pre-specified arrival rates has arrival rate λ_0 when the queue is empty and λ_1 when the queue is busy. □

Having established Assumptions 1 to 3, global attraction of the unique fixed point follows from Theorem 5.3 and asymptotic insensitivity within the class of hyperexponential distributions of order 2 follows for the traditional push strategy by repeating the arguments in the proof of Theorem 7.1.

9 CONCLUSIONS

In this paper we established an asymptotic insensitivity result for the supermarket model with processor sharing servers for the class of hyperexponential distributions of order 2. To the best of our knowledge, it is the first result of its kind for systems with PS service. More specifically, we showed that the weak limit of the stationary distributions as the number of servers tends to infinity is given by the Dirac measure of a fixed point, the queue length distribution of which is insensitive to the job size distribution. The main step in proving this result is showing that the set of ODEs describing the evolution of the mean field limit, has a global attractor. We also demonstrated, using the traditional push strategy in distributed systems, that our results can be of use to prove asymptotic insensitivity results beyond the supermarket model.

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A SERVICE COMPLETIONS IN PHASE 2: GENERAL CASE

The expression in (24) for (22) is only valid in case $i_{k+1} \geq i_k + 2$, for $k = 0, \ldots, s$. This is for instance needed for $g^{j+1}_{i_1,\ldots,i_k,i_k+1,i_{k+1},\ldots,i_s}$ to be well defined. In this Appendix we derive a general expression for (22), where $i_{k+1} = i_k + 1$ for some k values is allowed and combine this expression with (23) (which was shown to be equivalent to (22)), to conclude that the sum of the terms corresponding to service completions in phase 1 and 2 together are monotone when $\nu_1 > \nu_2$. For ease of presentation we once more suppress the dependence on $h(t)$.

For given $j, s, i_1, \ldots, i_s$ as in Definition 4.2, with $i_s < \infty$ and $i_{s+1} = \infty$, we define inductively $d_1 = 1$ and $d_k = 1 + 1[i_k - i_{k-1} = 1]d_{k-1}$ for $k = 2, \ldots, s$. We further define an injection $\sigma : \{1, \ldots, \tilde{s}\} \to \{1, \ldots, s\}$ as follows: $\sigma(\kappa)$ is the κ-th index k such that $i_{k+1} - i_k \geq 2$, not counting whether or not $i_1 \geq 2$. As $i_{s+1} = \infty$, we have $\tilde{s} \geq 1$ and $\sigma(\tilde{s}) = s$. We also set $\sigma(0) = 0$. We now prove three lemmas which are combined afterwards.

Lemma A.1. *Define the following formulas, these are illustrated in Figure 4*

$$S = \sum_{k=0}^{\sigma(1)-1} \sum_{i=i_k}^{i_{k+1}-1} w_{i,j-k} \frac{j-k}{i+j-k} + w_{i_{\sigma(1)},j-\sigma(1)} \frac{j-\sigma(1)}{i_{\sigma(1)} + j - \sigma(1)},$$

for $\kappa \in \{1, \ldots, \tilde{s}\}$*:*

$$T^{(\sigma(\kappa))} = \sum_{i=i_{\sigma(\kappa)}+1}^{i_{\sigma(\kappa)+1}-2} w_{i,j-\sigma(\kappa)} \frac{j-\sigma(\kappa)}{i+j-\sigma(\kappa)},$$

and for $\kappa \in \{1, \ldots, \tilde{s}-1\}$*:*

$$U^{(\sigma(\kappa))} = w_{i_{\sigma(\kappa)+1}-1,j-\sigma(\kappa)} \frac{j-\sigma(\kappa)}{i_{\sigma(\kappa)+1} - 1 + j - \sigma(\kappa)}$$

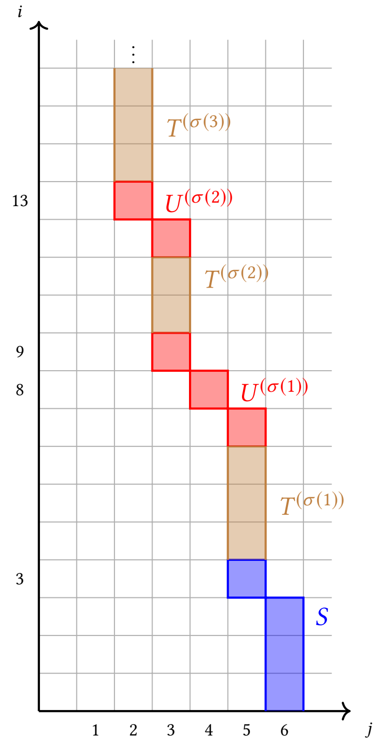

Fig. 4. Illustration of the formulas defined in Lemma A.1 for $g^6_{3,8,9,13}(h(t))$, we have $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 4$.

$$+\sum_{k=\sigma(\kappa)+1}^{\sigma(\kappa+1)-1}\sum_{i=i_k}^{i_{k+1}-1} w_{i,j-k}\frac{j-k}{i+j-k} + w_{i_{\sigma(\kappa+1)},j-\sigma(\kappa+1)}\frac{j-\sigma(\kappa+1)}{i_{\sigma(\kappa+1)}+j-\sigma(\kappa+1)}.$$

For ease of notation set for $k \notin \sigma(\{1,\ldots,\tilde{s}\})$:

$$T^{(k)} = 0,$$

and for $k \notin \sigma(\{1,\ldots,\tilde{s}-1\})$:

$$U^{(k)} = 0.$$

Then:

$$-\nu_2\sum_{k=0}^{s}\sum_{i=i_k}^{i_{k+1}-1} w_{i,j-k}\frac{j-k}{i+j-k} = -\nu_2(S + \sum_{k=1}^{s} T^{(k)} + \sum_{k=1}^{s} U^{(k)}), \tag{38}$$

further,

$$S = g^j_{i_1,\ldots,i_s} - \sum_{i=1}^{i_1-1} g^{j+1}_{i,i_1,\ldots,i_s}\left(\frac{j}{i+j-1} - \frac{j}{i+j}\right)$$
$$- g^{j+1}_{i_1,\ldots,i_{\sigma(1)},i_{\sigma(1)}+1,i_{\sigma(1)+1},\ldots,i_s}\frac{j}{i_1-1+j} - \sum_{k=1}^{\sigma(1)} w_{i_k,j-k}\frac{d_k}{i_k+j-k},$$

for $\kappa \in \{1, \ldots, \tilde{s}\}$*:*

$$
\begin{aligned}
T^{(\sigma(\kappa))} &= g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i_{\sigma(\kappa)}+1,i_{\sigma(\kappa)+1},\ldots,i_s} \frac{j-\sigma(\kappa)}{i_{\sigma(\kappa)}+j-\sigma(\kappa)} \\
&\quad - \sum_{i=i_{\sigma(\kappa)}+1}^{i_{\sigma(\kappa)+1}-1} g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i,i_{\sigma(\kappa)+1},\ldots,i_s} \left(\frac{j-\sigma(\kappa)}{i+j-\sigma(\kappa)-1} - \frac{j-\sigma(\kappa)}{i+j-\sigma(\kappa)} \right) \\
&\quad - 1[\kappa < \tilde{s}] g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i_{\sigma(\kappa)+1}-1,i_{\sigma(\kappa)+1},\ldots,i_s} \frac{j-\sigma(\kappa)}{i_{\sigma(\kappa)+1}-1+j-\sigma(\kappa)},
\end{aligned}
$$

and for $\kappa \in \{1, \ldots, \tilde{s}-1\}$*:*

$$
\begin{aligned}
U^{(\sigma(\kappa))} &= g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i_{\sigma(\kappa)+1}-1,i_{\sigma(\kappa)+1},\ldots,i_s} \frac{j-\sigma(\kappa)}{i_{\sigma(\kappa)+1}-1+j-\sigma(\kappa)} \\
&\quad - g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa+1)},i_{\sigma(\kappa+1)}+1,i_{\sigma(\kappa+1)+1},\ldots,i_s} \frac{j-\sigma(\kappa)}{i_{\sigma(\kappa)+1}-1+j-\sigma(\kappa)} - \sum_{k=\sigma(\kappa)+1}^{\sigma(\kappa+1)} w_{i_k,j-k} \frac{d_k}{i_k+j-k}.
\end{aligned}
$$

Proof. By writing

$$
\begin{aligned}
S + \sum_{k=1}^{s} T^{(k)} + \sum_{k=1}^{s} U^{(k)} &= S + \sum_{\kappa=1}^{\tilde{s}} T^{(\sigma(\kappa))} + \sum_{\kappa=1}^{\tilde{s}-1} U^{(\sigma(\kappa))} \\
&= S + T^{(\sigma(1))} + U^{(\sigma(1))} + \cdots + T^{(\sigma(\tilde{s}-1))} + U^{(\sigma(\tilde{s}-1))} + T^{(\sigma(\tilde{s}))},
\end{aligned}
$$

the first claim clearly holds. We have for $\kappa \in \{1, \ldots, \tilde{s}\}$

$$
\begin{aligned}
T^{(\sigma(\kappa))} &= \sum_{i=i_{\sigma(\kappa)}+1}^{i_{\sigma(\kappa)+1}-2} \left(g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i} - g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i+1} \right) \frac{j-\sigma(\kappa)}{i+j-\sigma(\kappa)} \\
&= \sum_{i=i_{\sigma(\kappa)}+1}^{i_{\sigma(\kappa)+1}-2} \left(g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i,i_{\sigma(\kappa)+1},\ldots,i_s} - g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i+1,i_{\sigma(\kappa)+1},\ldots,i_s} \right) \frac{j-\sigma(\kappa)}{i+j-\sigma(\kappa)},
\end{aligned}
$$

where we relied on (6) for the first equality and (5) for the second. By adding and subtracting zero to the sums, we find that this is equal to (where the indicator function is due to $i_{\sigma(\tilde{s})+1} = i_{s+1} = \infty$)

$$
\begin{aligned}
& g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i_{\sigma(\kappa)}+1,i_{\sigma(\kappa)+1},\ldots,i_s} \frac{j-\sigma(\kappa)}{i_{\sigma(\kappa)}+j-\sigma(\kappa)} - \sum_{i=i_{\sigma(\kappa)}}^{i_{\sigma(\kappa)+1}-2} g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i+1,i_{\sigma(\kappa)+1},\ldots,i_s} \frac{j-\sigma(\kappa)}{i+j-\sigma(\kappa)} \\
& + \sum_{i=i_{\sigma(\kappa)}+1}^{i_{\sigma(\kappa)+1}-1} g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i,i_{\sigma(\kappa)+1},\ldots,i_s} \frac{j-\sigma(\kappa)}{i+j-\sigma(\kappa)} - g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i_{\sigma(\kappa)+1}-1,i_{\sigma(\kappa)+1},\ldots,i_s} \frac{1[\kappa < \tilde{s}](j-\sigma(\kappa))}{i_{\sigma(\kappa)+1}-1+j-\sigma(\kappa)} \\
& = g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i_{\sigma(\kappa)}+1,i_{\sigma(\kappa)+1},\ldots,i_s} \frac{j-\sigma(\kappa)}{i_{\sigma(\kappa)}+j-\sigma(\kappa)} \\
& - \sum_{i=i_{\sigma(\kappa)}+1}^{i_{\sigma(\kappa)+1}-1} g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i,i_{\sigma(\kappa)+1},\ldots,i_s} \left(\frac{j-\sigma(\kappa)}{i+j-\sigma(\kappa)-1} - \frac{j-\sigma(\kappa)}{i+j-\sigma(\kappa)} \right) \\
& - 1[\kappa < \tilde{s}] g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i_{\sigma(\kappa)+1}-1,i_{\sigma(\kappa)+1},\ldots,i_s} \frac{j-\sigma(\kappa)}{i_{\sigma(\kappa)+1}-1+j-\sigma(\kappa)}.
\end{aligned}
$$

This proves the expression for $T^{(\sigma(\kappa))}$. Suppose $i_1 = 1$, then by the definition of σ,

$$S = \sum_{k=0}^{\sigma(1)} w_{i_k,j-k} \frac{j-k}{i_k+j-k}.$$

The definition of σ implies that $i_k = k = d_k$ for $k = 1, \ldots, \sigma(1)$ when $i_1 = 1$. By using (7) and then (5), we get

$$\begin{aligned}
S &= g^j - g_1^{j+1} + \sum_{k=1}^{\sigma(1)} w_{i_k,j-k} \frac{j-k}{i_k+j-k} = g^j - g_1^{j+1} + \sum_{k=1}^{\sigma(1)} w_{i_k,j-k}(1 - \frac{k}{i_k+j-k}) \\
&= g^j_{i_1,\ldots,i_{\sigma(1)}} - g^{j+1}_{1,i_1+1,\ldots,i_{\sigma(1)}+1} - \sum_{k=1}^{\sigma(1)} w_{i_k,j-k} \frac{d_k}{i_k+j-k} \\
&= g^j_{i_1,\ldots,i_s} - g^{j+1}_{i_1,\ldots,i_{\sigma(1)},i_{\sigma(1)}+1,i_{\sigma(1)+1},\ldots,i_s} - \sum_{k=1}^{\sigma(1)} w_{i_k,j-k} \frac{d_k}{i_k+j-k}.
\end{aligned}$$

Suppose now $i_1 \geq 2$, then similarly as $i_k - k = i_1 - 1$ and $d_k = k$ for $k = 1, \ldots, \sigma(1)$

$$\begin{aligned}
S &= \sum_{i=0}^{i_1-1} w_{i,j} \frac{j}{i+j} + \sum_{k=1}^{\sigma(1)} w_{i_k,j-k} \frac{j-k}{i_k+j-k} \\
&= g^j - g_1^{j+1} + \sum_{i=1}^{i_1-1} w_{i,j} \frac{j}{i+j} + \sum_{k=1}^{\sigma(1)} w_{i_k,j-k} \frac{j-k}{i_k+j-k} \\
&= g^j_{i_1,\ldots,i_s} - g^{j+1}_{1,i_1,\ldots,i_s} + \sum_{i=1}^{i_1-2} \left(g^{j+1}_{i,i_1,\ldots,i_s} - g^{j+1}_{i+1,i_1,\ldots,i_s} \right) \frac{j}{i+j} \\
&+ \left(g^{j+1}_{i_1-1} - g^{j+1}_{i_1} \right) \frac{j}{i_1-1+j} + \sum_{k=1}^{\sigma(1)} w_{i_k,j-k}(\frac{j}{i_1-1+j} - \frac{k}{i_k+j-k}) \\
&= g^j_{i_1,\ldots,i_s} - g^{j+1}_{1,i_1,\ldots,i_s} + \sum_{i=1}^{i_1-2} \left(g^{j+1}_{i,i_1,\ldots,i_s} - g^{j+1}_{i+1,i_1,\ldots,i_s} \right) \frac{j}{i+j} \\
&+ \left(g^{j+1}_{i_1-1,i_1,\ldots,i_{\sigma(1)}} - g^{j+1}_{i_1,i_1+1,\ldots,i_{\sigma(1)}+1} \right) \frac{j}{i_1-1+j} - \sum_{k=1}^{\sigma(1)} w_{i_k,j-k} \frac{d_k}{i_k+j-k}.
\end{aligned}$$

Similarly to the proof of $T^{(k)}$, we find that S is equal to

$$\begin{aligned}
& g^j_{i_1,\ldots,i_s} - g^{j+1}_{1,i_1,\ldots,i_s} + g^{j+1}_{1,i_1,\ldots,i_s} - g^{j+1}_{i_1-1,i_1,\ldots,i_s} \frac{j}{i_1-1+j} \\
& - \sum_{i=1}^{i_1-1} g^{j+1}_{i,i_1,\ldots,i_s} \left(\frac{j}{i+j-1} - \frac{j}{i+j} \right) + \left(g^{j+1}_{i_1-1,i_1,\ldots,i_s} - g^{j+1}_{i_1,i_1+1,\ldots,i_{\sigma(1)}+1,i_{\sigma(1)+1},\ldots,i_s} \right) \frac{j}{i_1-1+j} \\
& - \sum_{k=1}^{\sigma(1)} w_{i_k,j-k} \frac{d_k}{i_k+j-k} \\
& = g^j_{i_1,\ldots,i_s} - \sum_{i=1}^{i_1-1} g^{j+1}_{i,i_1,\ldots,i_s} \left(\frac{j}{i+j-1} - \frac{j}{i+j} \right)
\end{aligned}$$

$$- g^{j+1}_{i_1,\ldots,i_{\sigma(1)},i_{\sigma(1)}+1,i_{\sigma(1)+1},\ldots,i_s} \frac{j}{i_1 - 1 + j} - \sum_{k=1}^{\sigma(1)} w_{i_k,j-k} \frac{d_k}{i_k + j - k}.$$

Finally, noting that $i_{k+1} - 1 = i_k$ for $k = \sigma(\kappa) + 1, \ldots, \sigma(\kappa+1) - 1$, $i_k - k = i_{\sigma(\kappa)+1} - \sigma(\kappa) - 1$ and $d_k = k - \sigma(\kappa)$ for $k = \sigma(\kappa) + 1, \ldots, \sigma(\kappa+1)$, we find similarly to the proof of S with $i_1 = 1$ for $\kappa \in \{1, \ldots, \tilde{s} - 1\}$:

$$\begin{aligned}
U^{(\sigma(\kappa))} &= w_{i_{\sigma(\kappa)+1}-1,j-\sigma(\kappa)} \frac{j - \sigma(\kappa)}{i_{\sigma(\kappa)+1} - 1 + j - \sigma(\kappa)} + \sum_{k=\sigma(\kappa)+1}^{\sigma(\kappa+1)} w_{i_k,j-k} \frac{j-k}{i_k + j - k} \\
&= \left(g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i_{\sigma(\kappa)+1}-1} - g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)+1}} \right) \frac{j - \sigma(\kappa)}{i_{\sigma(\kappa)+1} - 1 + j - \sigma(\kappa)} \\
&+ \sum_{k=\sigma(\kappa)+1}^{\sigma(\kappa+1)} w_{i_k,j-k} \left(\frac{j - \sigma(\kappa)}{i_{\sigma(\kappa)+1} + j - \sigma(\kappa) - 1} + \frac{\sigma(\kappa) - k}{i_k + j - k} \right) \\
&= g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i_{\sigma(\kappa)+1}-1,i_{\sigma(\kappa)+1},\ldots,i_{\sigma(\kappa+1)}} \frac{j - \sigma(\kappa)}{i_{\sigma(\kappa)+1} - 1 + j - \sigma(\kappa)} \\
&- g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)+1},i_{\sigma(\kappa)+1}+1,\ldots,i_{\sigma(\kappa+1)}+1} \frac{j - \sigma(\kappa)}{i_{\sigma(\kappa)+1} - 1 + j - \sigma(\kappa)} \\
&- \sum_{k=\sigma(\kappa)+1}^{\sigma(\kappa+1)} w_{i_k,j-k} \frac{d_k}{i_k + j - k} \\
&= g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa)},i_{\sigma(\kappa)+1}-1,i_{\sigma(\kappa)+1},\ldots,i_s} \frac{j - \sigma(\kappa)}{i_{\sigma(\kappa)+1} - 1 + j - \sigma(\kappa)} \\
&- g^{j+1}_{i_1,\ldots,i_{\sigma(\kappa+1)},i_{\sigma(\kappa+1)}+1,i_{\sigma(\kappa+1)+1},\ldots,i_s} \frac{j - \sigma(\kappa)}{i_{\sigma(\kappa)+1} - 1 + j - \sigma(\kappa)} \\
&- \sum_{k=\sigma(\kappa)+1}^{\sigma(\kappa+1)} w_{i_k,j-k} \frac{d_k}{i_k + j - k}.
\end{aligned}$$

This finishes the proof of the first lemma. □

Lemma A.2. *We have for* $\kappa = 0, \ldots, \tilde{s} - 1$

$$\frac{j - \sigma(\kappa)}{i_{\sigma(\kappa)+1} + j - (\sigma(\kappa) + 1)} \geq \frac{j - \sigma(\kappa+1)}{i_{\sigma(\kappa+1)} + j - \sigma(\kappa+1)}.$$

Proof. As $i_{\sigma(\kappa+1)} - \sigma(\kappa+1) \geq i_{\sigma(\kappa)+1} - (\sigma(\kappa) + 1)$, the claim holds if

$$\frac{j - \sigma(\kappa)}{i_{\sigma(\kappa+1)} + j - \sigma(\kappa+1)} \geq \frac{j - \sigma(\kappa+1)}{i_{\sigma(\kappa+1)} + j - \sigma(\kappa+1)}.$$

But this is true as $\sigma(\kappa+1) > \sigma(\kappa)$. □

Lemma A.3. *We have for every* $k = 1, \ldots, s-1$*: if* $\frac{d_{k+1}}{i_{k+1}+j-(k+1)} - \frac{d_k}{i_k+j-k} > 0$ *then*

$$\frac{d_{k+1}}{i_{k+1} + j - (k+1)} - \frac{d_k}{i_k + j - k} = \frac{i_{k+1}}{i_{k+1} + j - (k+1)} - \frac{i_k}{i_k + j - k}. \tag{39}$$

Proof. Suppose $d_{k+1} = 1$, i.e. $i_{k+1} - i_k \geq 2$. Then $i_k + j - k < i_{k+1} + j - (k+1)$ and thus

$$\frac{d_{k+1}}{i_{k+1} + j - (k+1)} - \frac{d_k}{i_k + j - k} < \frac{d_{k+1}}{i_{k+1} + j - (k+1)} - \frac{d_k}{i_{k+1} + j - (k+1)} = \frac{1 - d_k}{i_{k+1} + j - (k+1)} \leq 0.$$

Hence, it suffices prove the claim for the case where $d_{k+1} = 1 + d_k$, i.e. where $i_{k+1} - i_k = 1$. (39) is then equivalent to

$$\frac{d_{k+1}}{i_k + j - k} - \frac{d_k}{i_k + j - k} = \frac{i_{k+1}}{i_k + j - k} - \frac{i_k}{i_k + j - k},$$

which is true. ☐

We now combine these Lemmas. We can rewrite (38) using Lemma A.1. Note that the last term in $T^{(\sigma(\kappa))}$ cancels the first term in $U^{(\sigma(\kappa))}$, while the first term in $T^{(\sigma(\kappa))}$ can be combined with the last term in $U^{(\sigma(\kappa-1))}$ for $\kappa > 1$ and with S for $\kappa = 1$. Then, by Lemma A.2 all these terms have the desired sign and only the following sum remains to be dealt with

$$v_2 \sum_{k=1}^{s} w_{i_k,j-k} \frac{d_k}{i_k + j - k}. \tag{40}$$

Using Lemma 4.6, we can rewrite (40) as

$$\begin{aligned} v_2 \sum_{k=1}^{s} w_{i_k,j-k} \frac{d_k}{i_k + j - k} &= v_2 g^j_{i_1,\ldots,i_s} \frac{d_s}{i_s + j - s} \\ &- v_2 \sum_{k=0}^{s-1} g^j_{i_1,\ldots,i_k,i_{k+1}+1,\ldots,i_s+1} \left(\frac{d_{k+1}}{i_{k+1} + j - k - 1} - \frac{d_k}{i_k + j - k} \right). \end{aligned}$$

The coefficients in this sum may not have the desired sign, however, if this is the case they can be combined with the expression that was derived for (21), that is,

$$\begin{aligned} - v_1 \sum_{k=1}^{s} w_{i_k,j-k} \frac{i_k}{i_k + j - k} &= -v_1 g^j_{i_1,\ldots,i_s} \frac{i_s}{i_s + j - s} \\ &+ v_1 \sum_{k=0}^{s-1} g^j_{i_1,\ldots,i_k,i_{k+1}+1,\ldots,i_s+1} \left(\frac{i_{k+1}}{i_{k+1} + j - k - 1} - \frac{i_k}{i_k + j - k} \right). \end{aligned}$$

The proof is now finished using Lemma A.3 and $v_1 > v_2$.

B PROOF OF PROPOSITION 6.4

π^w is a fixed point of (28) if

$$\begin{aligned} 0 = {} & 1[i \geq 1] \lambda \left(\hat{\pi}^d_{i+j-1} - \hat{\pi}^d_{i+j} \right) \frac{\pi^w_{i-1,j}}{\hat{\pi}_{i+j-1} - \hat{\pi}_{i+j}} - \lambda \left(\hat{\pi}^d_{i+j} - \hat{\pi}^d_{i+j+1} \right) \frac{\pi^w_{i,j}}{\hat{\pi}_{i+j} - \hat{\pi}_{i+j+1}} \\ & + 1[j \geq 1] p_1 \mu_1 \pi^w_{i+1,j-1} \frac{i+1}{i+j} - \mu_1 \pi^w_{i,j} \frac{i}{i+j} - \mu_2 \pi^w_{i,j} \frac{j}{i+j} \\ & + \mu_2 \pi^w_{i,j+1} \frac{j+1}{i+j+1} + (1-p_1) \mu_1 \pi^w_{i+1,j} \frac{i+1}{i+j+1}. \end{aligned}$$

By using the definition of π^w, this is equivalent to

$$\begin{aligned} 0 = {} & 1[i \geq 1] \lambda (\hat{\pi}^d_{i+j-1} - \hat{\pi}^d_{i+j}) \binom{i+j-1}{i-1} \left(\frac{1}{\mu_1} \right)^{i-1} \left(\frac{p_1}{\mu_2} \right)^j - \lambda (\hat{\pi}^d_{i+j} - \hat{\pi}^d_{i+j+1}) \binom{i+j}{i} \left(\frac{1}{\mu_1} \right)^i \left(\frac{p_1}{\mu_2} \right)^j \\ & + 1[j \geq 1] p_1 \mu_1 (\hat{\pi}_{i+j} - \hat{\pi}_{i+j+1}) \binom{i+j}{i+1} \left(\frac{1}{\mu_1} \right)^{i+1} \left(\frac{p_1}{\mu_2} \right)^{j-1} \frac{i+1}{i+j} \\ & - \mu_1 (\hat{\pi}_{i+j} - \hat{\pi}_{i+j+1}) \binom{i+j}{i} \left(\frac{1}{\mu_1} \right)^i \left(\frac{p_1}{\mu_2} \right)^j \frac{i}{i+j} \end{aligned}$$

$$
\begin{aligned}
&- \mu_2(\hat{\pi}_{i+j} - \hat{\pi}_{i+j+1}) \binom{i+j}{i} \left(\frac{1}{\mu_1}\right)^i \left(\frac{p_1}{\mu_2}\right)^j \frac{j}{i+j} \\
&+ \mu_2(\hat{\pi}_{i+j+1} - \hat{\pi}_{i+j+2}) \binom{i+j+1}{i} \left(\frac{1}{\mu_1}\right)^i \left(\frac{p_1}{\mu_2}\right)^{j+1} \frac{j+1}{i+j+1} \\
&+ (1-p_1)\mu_1(\hat{\pi}_{i+j+1} - \hat{\pi}_{i+j+2}) \binom{i+j+1}{i+1} \left(\frac{1}{\mu_1}\right)^{i+1} \left(\frac{p_1}{\mu_2}\right)^j \frac{i+1}{i+j+1}.
\end{aligned}
$$

We can rewrite this as

$$
\begin{aligned}
0 = &\lambda(\hat{\pi}^d_{i+j-1} - \hat{\pi}^d_{i+j}) \frac{\mu_1 i}{i+j} - \lambda(\hat{\pi}^d_{i+j} - \hat{\pi}^d_{i+j+1}) + \mu_2(\hat{\pi}_{i+j} - \hat{\pi}_{i+j+1}) \frac{j}{i+j} - \mu_1(\hat{\pi}_{i+j} - \hat{\pi}_{i+j+1}) \frac{i}{i+j} \\
&- \mu_2(\hat{\pi}_{i+j} - \hat{\pi}_{i+j+1}) \frac{j}{i+j} + p_1(\hat{\pi}_{i+j+1} - \hat{\pi}_{i+j+2}) + (1-p_1)(\hat{\pi}_{i+j+1} - \hat{\pi}_{i+j+2}),
\end{aligned}
$$

or, equivalently,

$$
0 = (\lambda(\hat{\pi}^d_{i+j-1} - \hat{\pi}^d_{i+j}) - (\hat{\pi}_{i+j} - \hat{\pi}_{i+j+1})) \frac{\mu_1 i}{i+j} - (\lambda(\hat{\pi}^d_{i+j} - \hat{\pi}^d_{i+j+1}) - (\hat{\pi}_{i+j+1} - \hat{\pi}_{i+j+2})). \tag{41}
$$

But this is true by definition of $\hat{\pi}$.

C PROOF OF PROPOSITION 8.2

By (33), $F_1^1(h)$ is only decreasing in $g_1^1 = h_{1,0}$. So we may assume that the indices $\{j, i_1, \ldots, i_s\}$ are different from $j = 1, i_1 = 1$. We then have

$$
\begin{aligned}
F^j_{i_1,\ldots,i_s}(h) &= f_{0,j}(h) + \sum_{k=1}^{s} (f_{i_k,j-k}(h) - f_{i_k-1,j-k+1}(h)) \\
&= \lambda h_{1,0} \sum_{k=1}^{s} w_{i_k-1,j-k}(h).
\end{aligned}
$$

The arrivals can be visualised in the same way as those for the supermarket model (see Figure 3). If $s = 0$ then there is nothing to show, so suppose $s \geq 1$. We need to check two cases: $i_1 = 1$ and $i_1 > 1$. Set $c_k = \lambda h_{1,0} = \lambda g_1^1(h)$ for $k = 1, \ldots, s$ and $c_0 = c_{s+1} = 0$. Suppose first $i_1 > 1$. By using Lemma 4.6, we get

$$
\begin{aligned}
F^j_{i_1,\ldots,i_s}(h) &= \sum_{k=1}^{s} c_k w_{i_k-1,j-k}(h) \\
&= g^j_{i_1-1,\ldots,i_s-1}(h) c_s - \sum_{k=0}^{s-1} g^j_{i_1-1,\ldots,i_k-1,i_{k+1},\ldots,i_s}(h) \Big(c_{k+1} - c_k\Big) \\
&= \lambda g_1^1(h)(g^j_{i_1-1,\ldots,i_s-1}(h) - g^j_{i_1,\ldots,i_s}(h)).
\end{aligned}
$$

If $h, \tilde{h} \in \Omega_B$ such that $h \leq_C \tilde{h}$ and $g^j_{i_1,\ldots,i_s}(h) = g^j_{i_1,\ldots,i_s}(\tilde{h})$, then $g_1^1(h) \leq g_1^1(\tilde{h})$ and $g^j_{i_1-1,\ldots,i_s-1}(h) \leq g^j_{i_1-1,\ldots,i_s-1}(\tilde{h})$ such that

$$
\lambda g_1^1(h)(g^j_{i_1-1,\ldots,i_s-1}(h) - g^j_{i_1,\ldots,i_s}(h)) \leq \lambda g_1^1(\tilde{h})(g^j_{i_1-1,\ldots,i_s-1}(\tilde{h}) - g^j_{i_1,\ldots,i_s}(\tilde{h})).
$$

Suppose now $i_1 = 1$. Proceeding similarly as in Proposition 6.2, we get

$$
F^j_{i_1,\ldots,i_s}(h) = \sum_{k=1}^{s} c_k w_{i_k-1,j-k}(h)
$$

$$= -c_1 g^j_{i_1,\ldots,i_s}(h) + \sum_{k=1}^{s} g^{j-1}_{i_2-1,\ldots,i_k-1,i_{k+1},\ldots,i_s}(h)\,(c_k - c_{k+1})$$
$$= \lambda g^1_1(h)(g^{j-1}_{i_2-1,\ldots,i_s-1}(h) - g^j_{i_1,\ldots,i_s}(h)).$$

If $h, \tilde{h} \in \Omega_B$ such that $h \leq_C \tilde{h}$ and $g^j_{i_1,\ldots,i_s}(h) = g^j_{i_1,\ldots,i_s}(\tilde{h})$, then $g^1_1(h) \leq g^1_1(\tilde{h})$ and $g^{j-1}_{i_2-1,\ldots,i_s-1}(h) \leq g^{j-1}_{i_2-1,\ldots,i_s-1}(\tilde{h})$ such that

$$\lambda g^1_1(h)(g^{j-1}_{i_2-1,\ldots,i_s-1}(h) - g^j_{i_1,\ldots,i_s}(h)) \leq \lambda g^1_1(\tilde{h})(g^{j-1}_{i_2-1,\ldots,i_s-1}(\tilde{h}) - g^j_{i_1,\ldots,i_s}(\tilde{h})).$$

This finishes the proof.