

ADI finite difference schemes for the Heston model with correlation

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Option pricing in the Heston model

European call option gives the holder the right to buy a given asset at a prescribed *maturity* date T for a prescribed *strike* price K .

Let S_t denote the value of the asset at time $t \geq 0$.

The *payoff* of the call option is $\max(0, S_T - K)$.

For the evolution of S_t we consider the popular [Heston stochastic volatility model](#) (1993):

$$\begin{cases} dS_t &= r_{df} S_t dt + \sqrt{V_t} S_t dW_t^1, \\ dV_t &= \kappa(\eta - V_t) dt + \sigma\sqrt{V_t} dW_t^2 \end{cases}$$

with real parameters $\kappa, \eta, \sigma, r_{df}$.

W_t^1, W_t^2 are Brownian motions with [correlation factor](#) $\rho \in [-1, 1]$.

Let $u(s, v, t)$ be the fair price of the call option if $S_{T-t} = s$, $V_{T-t} = v$.

Financial option pricing theory yields that u satisfies a parabolic PDE,

$$\frac{\partial u}{\partial t} = \frac{1}{2} s^2 v \frac{\partial^2 u}{\partial s^2} + \rho \sigma s v \frac{\partial^2 u}{\partial s \partial v} + \frac{1}{2} \sigma^2 v \frac{\partial^2 u}{\partial v^2} + r_d s \frac{\partial u}{\partial s} + \kappa(\eta - v) \frac{\partial u}{\partial v} - r_d u$$

for $0 < t \leq T$, $s > 0$, $v > 0$. We call this the **Heston PDE**.

The payoff gives the initial condition

$$u(s, v, 0) = \max(0, s - K).$$

Further, a boundary condition at $s = 0$ holds,

$$u(0, v, t) = 0.$$

The above equations constitute an initial-boundary value problem for a time-dependent convection-diffusion-reaction equation on an unbounded, two-dimensional spatial domain.

If the correlation $\rho \neq 0$, then there is a mixed-derivative term,

$$\frac{\partial^2 u}{\partial s \partial v}.$$

The Heston PDE forms a prototype for the many extensions of the well-known Black-Scholes PDE (1973) that arise in contemporary option pricing. These extensions are often comprised of parabolic PDEs in multiple space dimensions.

In finance there is a big demand for efficient, stable and robust codes for numerically solving such PDEs.



Semi-discretization Heston problem

To render the numerical solution of the Heston PDE feasible, first choose a bounded spatial domain $[0, S] \times [0, V]$ and appropriate additional boundary conditions.

We semi-discretize the PDE by replacing all spatial derivatives with suitable finite differences (FD).

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be any given function and $x_i = i \cdot \Delta x$ ($i \in \mathbb{Z}$), $\Delta x > 0$.

We deal with the following three FD formulas for the first derivative:

$$f'(x_i) \approx \left[\frac{1}{2} f_{i-2} - 2 f_{i-1} + \frac{3}{2} f_i \right] / \Delta x,$$

$$f'(x_i) \approx \left[-\frac{1}{2} f_{i-1} + \frac{1}{2} f_{i+1} \right] / \Delta x,$$

$$f'(x_i) \approx \left[-\frac{3}{2} f_i + 2 f_{i+1} - \frac{1}{2} f_{i+2} \right] / \Delta x.$$

These formulas are applied in the case of $\partial u / \partial s$ and $\partial u / \partial v$.

For the second derivative we take

$$f''(x_i) \approx [f_{i-1} - 2f_i + f_{i+1}] / (\Delta x)^2.$$

This FD formula is used for $\partial^2 u / \partial s^2$ and $\partial^2 u / \partial v^2$.

Next, suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $y_j = j \cdot \Delta y$ ($j \in \mathbb{Z}$), $\Delta y > 0$.

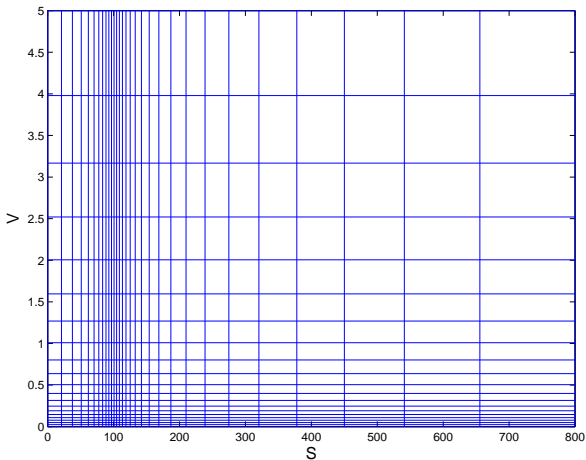
Then for the mixed derivative we consider

$$\frac{\partial^2 f}{\partial x \partial y}(x_i, y_j) \approx$$
$$\left[\frac{1}{4} f_{i-1, j-1} - \frac{1}{4} f_{i-1, j+1} - \frac{1}{4} f_{i+1, j-1} + \frac{1}{4} f_{i+1, j+1} \right] / (\Delta x \Delta y).$$

This FD formula is applied to $\partial^2 u / \partial s \partial v$.

All FD formulas above have a second-order truncation error.

In our actual application, we deal with a non-uniform grid in the (s, v) -domain such that many grid points lie near $(s, v) = (K, 0)$:

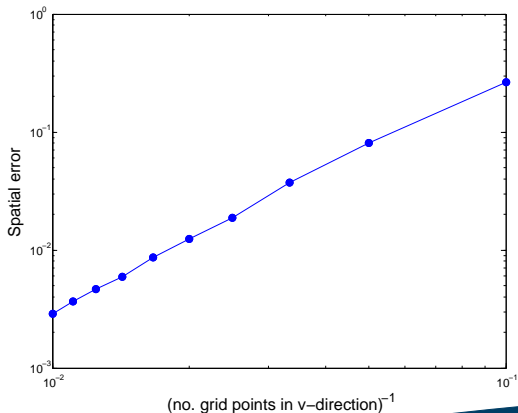


Semi-discretization error:

Comparison with code S. Foulon, KBC Bank. Data Bloomberg (2005):

$\kappa = 3, \eta = 0.12, \sigma = 0.04, \rho = 0.6, r_d = 0.01, r_f = 0.04, T = 1, K = 100.$

[no. of grid points s -direction] = $2 \times$ [no. of grid points v -direction]





Time integration semi-discrete Heston problem

FD discretization of the Heston problem yields an initial value problem for a large system of stiff ordinary differential equations (ODEs) of the form

$$U'(t) = A U(t) + b(t) \quad (0 < t \leq T), \quad U(0) = u_0$$

with given, fixed matrix A and vectors $b(t)$, u_0 .

Standard implicit numerical methods such as the trapezoidal rule (Crank–Nicolson) are often not effective.

For the numerical time integration of the ODE system, we consider splitting schemes of the **Alternating Direction Implicit (ADI)** type.

Splitting:

$$A = A_0 + A_1 + A_2$$

where

- ▶ A_0 corresponds to $\partial^2 u / \partial s \partial v$ term (!)
- ▶ A_1 corresponds to $\partial u / \partial s$, $\partial^2 u / \partial s^2$ terms
- ▶ A_2 corresponds to $\partial u / \partial v$, $\partial^2 u / \partial v^2$ terms

Assume $b(t) \equiv 0$. Let $\Delta t > 0$ and grid points $t_n = n \cdot \Delta t$.

Four ADI schemes yielding $U_n \approx U(t_n)$ ($n = 1, 2, 3, \dots$):

Douglas (Do) scheme

$$\begin{cases} Y_0 = U_{n-1} + \Delta t A U_{n-1}, \\ Y_j = Y_{j-1} + \theta \Delta t A_j (Y_j - U_{n-1}) \quad (j = 1, 2), \\ U_n = Y_2. \end{cases}$$

Parameter $\theta > 0$. Classical order (for general A_0, A_1, A_2) is 1.

Craig–Sneyd (CS) scheme

$$\left\{ \begin{array}{l} Y_0 = U_{n-1} + \Delta t A U_{n-1}, \\ Y_j = Y_{j-1} + \theta \Delta t A_j (Y_j - U_{n-1}) \quad (j = 1, 2), \\ \tilde{Y}_0 = Y_0 + \frac{1}{2} \Delta t A_0 (Y_2 - U_{n-1}), \\ \tilde{Y}_j = \tilde{Y}_{j-1} + \theta \Delta t A_j (\tilde{Y}_j - U_{n-1}) \quad (j = 1, 2), \\ U_n = \tilde{Y}_2. \end{array} \right.$$

Parameter $\theta > 0$.

Classical order is **2** iff $\theta = \frac{1}{2}$.

Modified Craig–Sneyd (MCS) scheme

$$\left\{ \begin{array}{l} Y_0 = U_{n-1} + \Delta t A U_{n-1}, \\ Y_j = Y_{j-1} + \theta \Delta t A_j (Y_j - U_{n-1}) \quad (j = 1, 2), \\ \hat{Y}_0 = Y_0 + \theta \Delta t A_0 (Y_2 - U_{n-1}), \\ \tilde{Y}_0 = \hat{Y}_0 + (\frac{1}{2} - \theta) \Delta t A (Y_2 - U_{n-1}), \\ \tilde{Y}_j = \tilde{Y}_{j-1} + \theta \Delta t A_j (\tilde{Y}_j - U_{n-1}) \quad (j = 1, 2), \\ U_n = \tilde{Y}_2. \end{array} \right.$$

Parameter $\theta > 0$.

Classical order is **2** for **all** θ .

Hundsdorfer–Verwer (HV) scheme

$$\left\{ \begin{array}{l} Y_0 = U_{n-1} + \Delta t A U_{n-1}, \\ Y_j = Y_{j-1} + \theta \Delta t A_j (Y_j - U_{n-1}) \quad (j = 1, 2), \\ \tilde{Y}_0 = Y_0 + \frac{1}{2} \Delta t A (Y_2 - U_{n-1}), \\ \tilde{Y}_j = \tilde{Y}_{j-1} + \theta \Delta t A_j (\tilde{Y}_j - Y_2) \quad (j = 1, 2), \\ U_n = \tilde{Y}_2. \end{array} \right.$$

Parameter $\theta > 0$.

Classical order is 2 for all θ .

References

- ▶ Peaceman & Rachford (1955)
- ▶ Douglas & Rachford (1956)
- ▶ Brian (1961)
- ▶ Douglas (1962)
- ▶ McKee & Mitchell (1970)
- ▶ Van der Houwen & Verwer (1979)
- ▶ Craig & Sneyd (1988)
- ▶ McKee, Wall & Wilson (1996)
- ▶ Hundsdoerfer (1999, 2002)
- ▶ Lanser, Blom & Verwer (2001)
- ▶ Hundsdoerfer & Verwer (2003)
- ▶ In 't Hout & Welfert (2007, 2009)
- ▶ In 't Hout & Foulon (2010)

Unconditional stability results - in von Neumann sense - for ADI schemes when applied to FD discretizations of two-dimensional convection-diffusion problems **with mixed derivative term**:

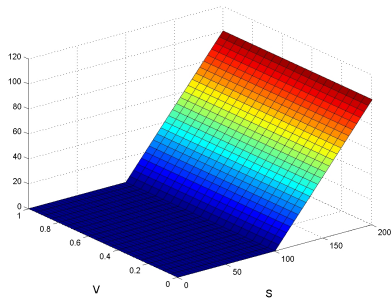
- ▶ McKee, Wall & Wilson ('96):
 - Do scheme is stable if $\theta = \frac{1}{2}$
- ▶ Craig & Sneyd ('88):
 - CS scheme is stable if $\theta = \frac{1}{2}$ and no convection
- ▶ In 't Hout & Welfert ('07, '09):
 - CS scheme is stable if $\theta = \frac{1}{2}$
 - MCS scheme is stable if $\theta \geq \frac{1}{3}$ and no convection
 - HV scheme is stable if $\theta \geq 1 - \frac{1}{2}\sqrt{2}$ and no convection
 - HV scheme is stable if $\theta \geq \frac{1}{2} + \frac{1}{6}\sqrt{3}$: *conjecture*



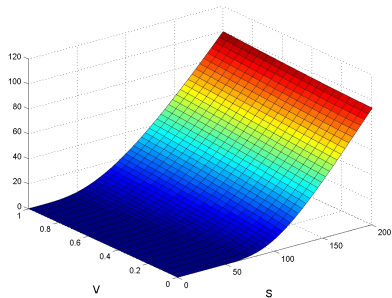
Numerical experiments

Bloomberg data

$\kappa = 3, \eta = 0.12, \sigma = 0.04, \rho = 0.6, r_d = 0.01, r_f = 0.04, T = 1, K = 100$



$t = 0$



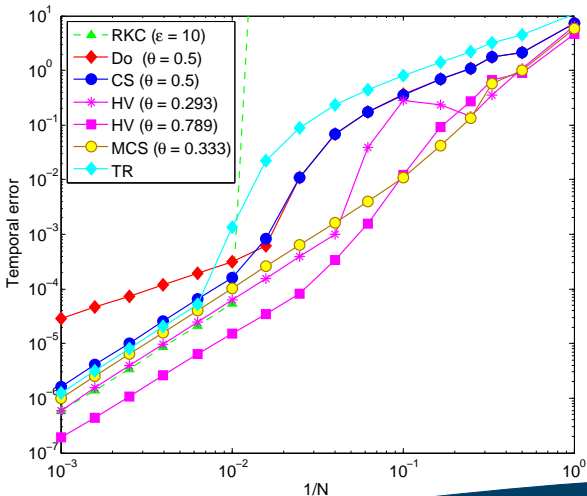
$t = T$

Numerical time-stepping schemes

- ▶ Do with $\theta = \frac{1}{2}$
- ▶ CS with $\theta = \frac{1}{2}$
- ▶ MCS with $\theta = \frac{1}{3}$
- ▶ HV with $\theta = 1 - \frac{1}{2}\sqrt{2}$ (HV1)
- ▶ HV with $\theta = \frac{1}{2} + \frac{1}{6}\sqrt{3}$ (HV2)
- ▶ Trapezoidal rule (TR)
- ▶ Runge–Kutta–Chebyshev (RKC)

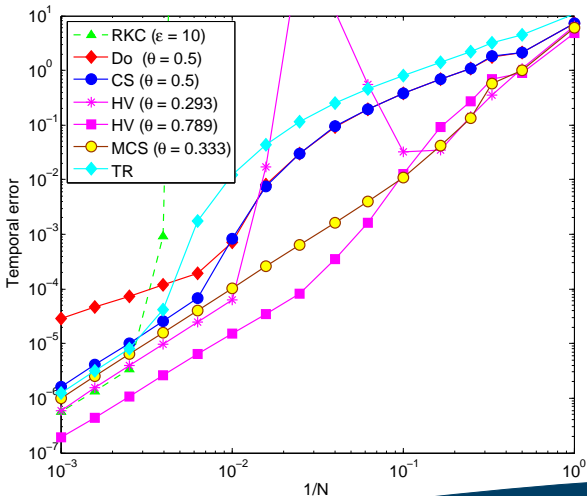
Number of grid points (s, v): 100×50 .

Global temporal errors of ADI schemes w.r.t. ODE system:



Number of grid points (s, v): 200×100 .

Global temporal errors of ADI schemes w.r.t. ODE system:



Observations

- ▶ TR: time-consuming.
- ▶ RKC, HV1: show instability, due to $\sigma \approx 0$.
- ▶ Do, CS, MCS, HV2, TR: show unconditional stability.
- ▶ Do, CS, TR: large errors for modest Δt , damping required.
- ▶ Do with damping: order of convergence 1.
- ▶ CS, TR with damping: order of convergence 2.
- ▶ MCS, HV2: order of convergence 2.



Conclusions and future research

Conclusions

- ▶ MCS with $\theta = \frac{1}{3}$ and HV with $\theta = \frac{1}{2} + \frac{1}{6}\sqrt{3}$ seem preferable.
- ▶ CS with $\theta = \frac{1}{2}$ and damping seems good second choice.

Current / future research

- ▶ Application of ADI FD approach to exotic options and higher-dimensional asset price models.
- ▶ Theoretical stability and convergence analysis.
- ▶ Practical implementation and experiments.
- ▶ Special features of option price models.
- ▶ Calibration, Greeks, ...