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RATIONAL INVARIANTS OF QUIVERS AND
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Abstract

Let V be a finite dimensional vectorspace over \mathbb{C} and let $\alpha = (\alpha(1), \dots, \alpha(m)) \in \mathbb{N}^m$. We say that the group $GL(\alpha) = \prod_{i=1}^m GL_{\alpha(i)}(\mathbb{C})$ acts Schurian on V if the stabilizer of a generic point is \mathbb{C}^* embedded diagonally in $GL(\alpha)$. It is shown that the field of rational invariants for this action $\mathbb{C}(V)^{GL(\alpha)}$ is stably rational to the field of rational n by n matrix invariants where $n = \gcd(\alpha(i) : 1 \leq i \leq m)$.

1. The problem

Throughout this note, we consider an algebraically closed field of characteristic zero and call it \mathbb{C} . Let G be an affine linear reductive group acting on a finite dimensional vectorspace almost freely, that is, the stabilizer of a generic point is trivial. One of the main open problems in invariant theory is to determine for which groups G the field of rational invariants $\mathbb{C}(V)^G$ is (stably) rational.

A first approach might be the following. Let G act almost freely on V , then there exists an affine G -invariant open subvariety U of V consisting of points with trivial stabilizer and we can form the quotient variety U/G . The canonical map from U to U/G is then a principal G -bundle in the étale topology and such objects are classified by elements of the group $H_{\text{ét}}^1(U/G, G)$. Therefore, if the G -bundle is trivialisable by a cover in the Zariski topology, we see that U is birational to $U/G \times G$ and since reductive groups are rational we see that U/G is stably rational.

Of course, being Zariski trivialisable is a fairly strong condition. Another well known argument concerning this problem is the so called 'no name lemma' (some say, no proof lemma) which gives us some freedom on the particular choice of the vectorspace as long as we are interested in stable rationality :

Lemma 1 : Let G be a reductive group acting on two vectorspaces V and V' such that the action on V is almost freely. Then the field of rational invariants of G on $V \oplus V'$ is rational over the field of rational invariants on V .

Proof : Let U be an affine open subvariety of V such that the map from U to U/G is a principal G -bundle in the étale topology. We form the vectorbundle $V' \times_G U \rightarrow U/G$ and note that the map from $V \times U$ to $V \times_G U$ is the quotient map for the action of G on $V \times U$.

Therefore, if G acts almost freely on both V and V' then the fields of rational invariants are stably rational over each other.

2. Some quiverology

Our main motivation comes from the study of representation of quivers. In this section we will briefly recall the setting. After the work of P. Gabriel [2] it became clear that several problems from linear algebra could be formulated and studied in a uniform way in the context of representations of quivers. A quiver Q is a quadruple (Q_0, Q_1, t, h) consisting of a finite set $Q_0 = \{1, \dots, m\}$ of vertices, a set Q_1 of arrows between these vertices and two maps $t, h : Q_1 \rightarrow Q_0$ assigning to an arrow ϕ its tail $t\phi$ and its head $h\phi$, respectively.

A representation V of a quiver Q is a family $\{V(i) : i \in Q_0\}$ of finite dimensional vector spaces over \mathbb{C} together with a family of linear maps $\{V(\phi) : V(t\phi) \rightarrow V(h\phi); \phi \in Q_1\}$. The m -tuple $\dim(V) = (\dim(V(i)))_{i \in Q_0} \in \mathbb{N}^m$ is called the dimension type of V . A morphism $f : V \rightarrow W$ between two representations is a family of linear maps $\{f_i : V(i) \rightarrow W(i); i \in Q_0\}$ such that for all arrows $\phi \in Q_1$ we have: $W(\phi) \circ f(t\phi) = f(h\phi) \circ V(\phi)$.

Given a dimension vector $\alpha = (\alpha(1), \dots, \alpha(m)) \in \mathbb{N}^m$ we define the representation space $R(Q, \alpha)$ to be the set of all representations V of Q such that $V(i) = \mathbb{C}^{\alpha(i)}$ for all $i \in Q_0$. Because $V \in R(Q, \alpha)$ is completely determined by the maps $V(\phi)$ we have that

$$R(Q, \alpha) = \bigoplus_{\phi \in Q_1} M_{\phi}(\mathbb{C})$$

where $M_{\phi}(\mathbb{C})$ denotes the \mathbb{C} -vector space of all $\alpha(h\phi)$ by $\alpha(t\phi)$ matrices with entries in \mathbb{C} .

We will consider the representation space $R(Q, \alpha)$ as an affine variety with coordinate ring $\mathbb{C}[Q, \alpha]$ and function field $\mathbb{C}(Q, \alpha)$. We have a canonical action of the linear reductive group $GL(\alpha) = \prod_{i=1}^m GL_{\alpha(i)}(\mathbb{C})$ on $R(Q, \alpha)$ by

$$(g.V)(\phi) = g_{h\phi} V g_{t\phi}^{-1}$$

for all $g = (g_1, \dots, g_n) \in GL(\alpha)$. The $GL(\alpha)$ -orbits in $R(Q, \alpha)$ are precisely the isomorphism classes of representations.

Ultimately, one is interested in the description of this orbit structure. It suffices clearly to restrict attention to indecomposable representations. V. Kac [3] conjectured that the variety parametrizing isoclasses of indecomposable α -representations of Q admits a finite cellular decomposition into locally closed subvarieties each isomorphic to some affine space. Unfortunately there is, at this moment, not much evidence to support this conjecture. An immediate consequence would be that the field of rational invariants (that is $\mathbb{C}(Q, \alpha)^{GL(\alpha)}$) is rational whenever α is a so called Schur root. Recall that α is said to be a Schur root if α -representations in general position are indecomposable, or equivalently, if there exists an α -representation with endomorphism ring reduced to \mathbb{C} .

Therefore, if we denote $PGL(\alpha) = GL(\alpha)/\mathbb{C}^*$ where \mathbb{C}^* is embedded diagonally in $GL(\alpha)$, Schur roots are precisely those dimension vectors α such that $PGL(\alpha)$ acts almost freely on the representation space $R(Q, \alpha)$. In the special case of the two loop quiver, (the classification of couples of n by n matrices under simultaneous conjugation), such a result would immediately imply the Merkurjev-Suslin result, the lifting problem for crossed products over local rings and the rationality of the moduli space of stable rank n bundles over the projective plane with Chern-numbers $c_1 = 0$ and $c_2 = n$. In this note we will show that this special case is really the heart of the problem. More precisely we will prove that if α is a Schur root for the quiver Q such that $\gcd(\alpha(i) : i \in Q_0) = n$, then the rational invariants are stably rational with the field of matrix invariants, that is the field of rational invariants of couples of n by n matrices under simultaneous conjugation with $GL_n(\mathbb{C})$. The Schur root assumption is no real restriction since Kac [4] has indicated how the rational invariants of an arbitrary dimension vector can be computed in terms of the rational invariants for the Schur roots occurring in the generic decomposition. Finally, we mention that C.M. Ringel has proved rationality of the rational invariants in case Q is a tame quiver.

2. Rational invariants and Azumaya algebras

In view of the no name lemma we may restrict our attention to the case that the vectorspace V is a representation space $R(Q, \alpha)$ where α is a Schur root for the quiver Q . Let U be the affine subvariety of $Rep(Q, \alpha)$ such that U is a principal $PGL(\alpha)$ - bundle over $U/PGL(\alpha)$. Remark that the functionfield of $U/PGL(\alpha)$ is the field of rational invariants.

In this case, we will show that one can obtain the cohomology class in $H_{\text{ét}}^1(X, PGL(\alpha))$ in a more concrete way in terms of an Azumaya algebra whose triviality is equivalent to the existence of a Zariski cover splitting the cohomology class.

An R -ring S is a ring with a specified homomorphism from R to S . If $K = \times_{i \in Q_0} \mathbb{C}$, then the path algebra $P(Q)$ of the quiver Q is a K -ring in the natural way. For a dimension vector α , the centre of $\times_{i \in Q_0} M_{\alpha(i)}(\mathbb{C})$ is K so we shall regard this as a K -ring; in turn, if $m = \sum_{i \in Q_0} \alpha(i)$, there is an embedding of $\times_{i \in Q_0} M_{\alpha(i)}(\mathbb{C})$ into $M_m(\mathbb{C})$ along the diagonal and we regard $M_m(\mathbb{C})$ as a K -ring via this embedding.

The group of automorphisms of $M_m(\mathbb{C})$ that fix K is isomorphic to $PGL(\alpha)$ since all automorphisms of $M_m(\mathbb{C})$ that fix the center are inner. Therefore, $H_{\text{ét}}^1(X, PGL(\alpha))$ classifies twisted forms of $M_m(\mathbb{C})$, that is, Azumaya algebras over X with a distinguished embedding of K that is split by an étale cover so that on the étale cover the embedding of K in matrices is conjugate to the original embedding defined by α . This allows us to deduce at once the following result.

Theorem 2 : Let α be a Schur root of the quiver Q such that $\gcd(\alpha(i) : i \in Q_0) = 1$. Then, the field of rational invariants for the dimension vector α is stably rational.

Proof : In this case, the Azumaya algebra must be split on a Zariski cover since the map from $K_0(K)$ to $K_0 M_m(\mathbb{C})$ is surjective and this forces the same to be true for the Azumaya algebra. Since the Azumaya algebra is split on a Zariski cover so is the principal $PGL(\alpha)$ - bundle $U \rightarrow U/PGL(\alpha)$, which implies that $Rep(Q, \alpha)$ is birational to $U/PGL(\alpha) \times PGL(\alpha)$. But $Rep(Q, \alpha)$ and $PGL(\alpha)$ are rational so $U/PGL(\alpha)$ and hence its functionfield is stably rational.

Let δ be an element of $H_{\text{ét}}^1(X, PGL(\alpha))$, let $U(\delta) \rightarrow X$ be the corresponding principal homogeneous space for $PGL(\alpha)$ in the étale topology and let $A(\delta)$ be the sheaf of Azumaya algebras over X . Then, an R -point of a commutative ring in $U(\delta)$ may be identified with a K -ring homomorphism from $R \otimes A(\delta)$ to $M_m(R)$. This allows one to identify the Azumaya algebra over $U/PGL(\alpha)$ associated to the canonical map $U \rightarrow U/PGL(\alpha)$ with the ring of matrix concomitants from U to $M_m(K)$.

Corollary 3 :

- (1) : Let α be a Schur root for the quiver Q and let Q' be a larger quiver on the same vertex set. Then, the field of rational invariants for α on Q' is rational over the field of rational invariants on Q .
- (2) : Let α be a Schur root for two quivers Q and Q' . Then, the field of rational invariants for α on Q is stably rational over the field of rational invariants on Q' . In particular, if α is a real Schur root for Q' , then the field of rational invariants for α on Q is stably rational.

Proof : (1) : Write $Rep(Q', \alpha) = Rep(Q, \alpha) \oplus W$ and apply the lemma. (2) : Apply the lemma twice. For the last part, recall that a real Schur root means that $R(Q', \alpha)$ has an open indecomposable orbit, whence the field of rational invariants is reduced to \mathbb{C} .

One final construction is needed. Let α be a Schur root for the quiver Q and let us denote $n = \gcd(\alpha(i) : i \in Q_0)$. We form a new quiver by adjoining one vertex 0 and an arrow from 0 to some vertex $i \in Q_0$. Let

α' be the extended dimension vector such that $\alpha'(0) = n$ and $\alpha'(i) = \alpha(i)$. Let U be an open subset of $Rep(Q, \alpha)$ such that there is an orbit map $U \rightarrow U/PGL(\alpha)$ which is a principal $PGL(\alpha)$ -bundle in the étale topology. Let $A(\alpha)$ be the associated Azumaya algebra. Because $gcd(\alpha(i)) = n$, we may assume by passing to a Zariski open subvariety if necessary that $A(\alpha)$ is isomorphic to $M_t(B(\alpha))$ where $m = tn$ and the embedding of K in $A(\alpha)$ may be refined to a set of matrix units. Let $\alpha(i) = sn$. Let W be the open subset of $Rep(Q^{ext}, \alpha')$ whose image in $Rep(Q, \alpha)$ is in U and where the new arrow is injective, and let $W/PGL(\alpha') \rightarrow U/PGL(\alpha)$ be the induced map. Then, $W/PGL(\alpha')$ represents rank one $B(\alpha)$ -submodules of free rank s $B(\alpha)$ -modules, which is a rational projective variety.

Theorem 4 : Let α be a Schur root for the quiver Q . Let $gcd(\alpha(i) : i \in Q_0) = n$. Let Q' be a quiver with one extra point 0 and at least one more arrow. Let $\alpha'(0) = n$ and $\alpha'(i) = \alpha(i)$ where defined. Then, the rational invariants for α' are rational over the rational invariants for α .

Proof : The last paragraph proves this for the case of precisely one arrow; then corollary 3 completes the proof.

We are now in a position to prove the main result of this note :

Theorem 5 : Let α be a Schur root for the quiver Q . Let $n = gcd(\alpha(i) : i \in Q_0)$. Then, the field of rational invariants for α is stably rational to the field of matrixinvariants for n by n matrices.

Proof : Consider the quiver Q' , a point point extension of Q where the number of arrows from the extra point 0 to the point i is $\alpha(i)/n$. Consider the open subvariety of $Rep(Q', \alpha')$ where the $\alpha(i)/n$ maps from $V(0)$ to $V(i)$ define an isomorphism from $V(0)^{\alpha(i)/n}$ to $V(i)$. This reduces the classification problem of the extended quiver to representations of the original quiver Q where each vertex space is in addition given a fixed representation as a vector space of the form $V^{\alpha(i)/n}$ where V is a vectorspace of dimension n . But this is the same as the classification of $\sum_{\phi \in Q_1} \alpha(t\phi)\alpha(h\phi)/n^2$ by n matrices up to simultaneous conjugation. This shows that the rational invariants for α' on Q' are stably rational to the rational invariants of $2n$ by n matrices under simultaneous conjugation. Finally, theorem 4 completes the proof.

One can describe the generic splitting field (or the Brauer-Severi variety) of the Azumaya algebra corresponding to the principal $PGL(\alpha)$ -bundle $U \rightarrow U/PGL(\alpha)$ in the following way. Take a one point extension of the quiver Q such that there is precisely one arrow pointing from the new vertex to any $i \in Q_0$ and take the dimension vector $\beta = (1, \alpha)$. Then, the field of rational invariants for β of this new quiver is the generic splitting field of the Azumaya algebra corresponding to the Schur root α of Q . Using theorem 2 we obtain that this generic splitting field is always stably rational. Another proof of this fact is : $(1, \alpha)$ is a real Schur root for the quiver on the vertices $0, i \in Q_0$ which has precisely $\alpha(i)$ arrows from 0 to i for all $i \in Q_0$ and no others. Then, by corollary 3 the rational invariants for $(1, \alpha)$ of the one point extension are stably rational.

4. Rational invariants and reflexion functors

We can give a purely representation theoretic proof of this result using the Bernstein-Gelfand-Ponomarev theory of reflection functors. Let $i \in Q_0$ be a sink, that is for no $\phi \in Q_1$ we have $t\phi = i$, and let

α be a dimension vector. We form a new quiver Q' by reversing the direction of the arrows connected to i and define a new dimension vector β by $\beta(j) = \alpha(j)$ if $j \neq i$ and $\beta(i) = \sum_{h\phi=i} \alpha(t\phi) - \alpha(i)$. Consider the open subvariety of $Rep(Q, \alpha)$

$$Rep'(Q, \alpha) = \{V \in Rep(Q, \alpha) : \oplus V(\phi) : \oplus_{h\phi=i} V(t\phi) \rightarrow V(i) \text{ is surjective} \}$$

And similarly we consider the open subvariety of $Rep(Q', \beta)$

$$Rep'(Q', \beta) = \{V \in Rep(Q', \beta) : \oplus V(\phi) : V(i) \rightarrow \oplus_{t\phi=i} V(h\phi) \text{ is injective} \}$$

then there exists a homeomorphism between $Rep'(Q, \alpha)/GL(\alpha)$ and $Rep'(Q', \beta)/GL(\beta)$ such that corresponding representations have isomorphic endomorphism rings. In particular if α is a Schur root for Q then β is a Schur root for Q' and then the rational invariants for α on Q are isomorphic to the rational invariants for β on Q' .

2nd Proof of theorem 5 : Let $i \in Q_0$ be such that $\alpha(i) = kn$ is minimal and let $j \in Q_0$ be such that $\alpha(j) = ln$ and k does not divide l , say $l = ak - b$ with $b \neq 0$. For a new quiver Q' on the same vertexset with a arrows pointing from i to j and all other arrows live on $Q_0 - j$ in such a way that α is a Schur root for Q' (can be done for example by trowing in lots of loops). By lemma 3 the rational invariants for α on Q are stably rational to those for α on Q' which are isomorphic to the rational invariants for β determined by $\beta(k) = \alpha(k)$, $k \neq j$ and $\beta(j) = bn < kn$ on the reflection of Q' in j . So, by induction we may assume that the rational invariants for α on Q are stably rational to those for γ on some quiver Q'' where $gcd(\gamma(i)) = n$ and $\gamma(i) = n$ for some $i \in Q_0$.

Now, we can proceed by induction on the number of vertices. Take $j \neq i$ such that $\gamma(j) = k.n$. Form a quiver Q^\dagger with precisely k arrows from i to j and the arrows on $Q_0^\dagger - j$ such that γ is a Schur root for Q^\dagger . Then, after applying reflexion with respect to the sink j we get a quiver with one vertex less s.t. the rational invariants are still stably rational to the original. Continuing in this way we will end up with the two loop quiver in i finishing the proof.

By applying the no name lemma we have therefore proved

THEOREM : Let $GL(\alpha)$ act Schurian on a vectorspace V . Then, the rational invariants $\mathcal{O}(V)^{GL(\alpha)}$ are stably rational to the field of rational matrixinvariants for n by n matrices where $n = gcd(\alpha(i) : 1 \leq i \leq m)$

We recall that E. Formanek proved rationality of the field of matrixinvariants for $n = 3$ and $n = 4$ ($n = 1$ or 2 is classical) and that D. Saltman proved retract rationality for all n squarefree. Hence, we obtain

Corollary 6 : Let α be a Schur root of a quiver Q such that $n = gcd(\alpha(i))$, then the rational invariants for α on Q are stably rational if $n \leq 4$ and are retract rational if n is squarefree.

Recall that the unramified Brauer group of a field L is the Brauer group of a smooth projective model of L . Saltman [6] proved that the unramified Brauer group of the field of matrixinvariants is reduced to the Brauer group of the basefield \mathbb{F} . As an immediate consequence of this result we have

Corollary 7 : Let α be a Schur root of a quiver Q . Then, the Brauer group of a projective smooth model of the rational invariants for α on Q is trivial.

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