NONCOMMUTATIVE GEOMETRY AND DUAL COALGEBRAS

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ABSTRACT. In [arXiv:math/0606241v2] M. Kontsevich and Y. Soibelman argue that the category of noncommutative (thin) schemes is equivalent to the category of coalgebras. We propose that under this correspondence the affine scheme \( \text{rep}(A) \) of a \( k \)-algebra \( A \) is the dual coalgebra \( A^o \) and draw some consequences. In particular, we describe the dual coalgebra \( A^o \) of \( A \) in terms of the \( A_{\infty} \)-structure on the Yoneda-space of all the simple finite dimensional \( A \)-representations.

CONTENTS

1. \( \text{rep}(A) = A^o \) ........................................ 1
2. \( \text{simp}(A) = \text{corad}(A^o) \) ......................... 2
3. the dual coalgebra \( A^o \) ................................ 3

References .................................................................. 5

1. \( \text{rep}(A) = A^o \)

Throughout, \( k \) will be a (commutative) field with separable closure \( \bar{k} \). In [3, §I.2] Maxim Kontsevich and Yan Soibelman define a noncommutative thin scheme to be a covariant functor commuting with finite projective limits

\[
X : \text{alg}^{fd}_k \longrightarrow \text{sets}
\]

from the category \( \text{alg}^{fd}_k \) of all finite dimensional \( k \)-algebras (associative with unit) to the category \( \text{sets} \) of all sets. They prove [3, Thm. 2.1.1] that every noncommutative thin scheme is represented by a \( k \)-coalgebra.

Recall that a \( k \)-coalgebra is a \( k \)-vectorspace \( C \) equipped with linear structural morphisms : a comultiplication \( \Delta : C \longrightarrow C \otimes C \) and a counit \( \epsilon : C \longrightarrow k \) satisfying the coassociativity \( (id \otimes \Delta) \Delta = (\Delta \otimes id) \Delta \) and counitary property \( (id \otimes \epsilon) \Delta = (\epsilon \otimes id) \Delta = id \).

By being representable they mean that every noncommutative thin scheme \( X \) has associated to it a \( k \)-coalgebra \( C_X \) with the property that for any finite dimensional \( k \)-algebra \( B \) there is a natural one-to-one correspondence

\[
X(B) = \text{alg}^d_k(B, C^*_X)
\]

Here, for a \( k \)-coalgebra \( C \) we denote by \( C^* \) the space of linear functionals \( \text{Hom}_k(C, k) \) which acquires a \( k \)-algebra structure by dualizing the structural coalgebra morphisms.

They call \( C_X \) the coalgebra of distributions on \( X \) and define the noncommutative algebra of functions on \( X \) to be the dual \( k \)-algebra \( k[X] = C^*_X \).

Whereas the dual \( C^* \) of a \( k \)-coalgebra is always a \( k \)-algebra, for a \( k \)-algebra \( A \) it is not true in general that the dual vectorspace \( A^* \) is a coalgebra, because \( (A \otimes A)^* \neq A^* \otimes A^* \).

Still, one can define the subspace

\[
A^o = \{ f \in A^* = \text{Hom}_k(A, k) \mid \text{ker}(f) \text{ contains a twosided ideal of finite codimension} \}
\]

and show that the duals of the structural morphisms on \( A \) determine a \( k \)-coalgebra structure on this dual coalgebra \( A^o \), see for example [5, Prop. 6.0.2].
With these definitions, Kostant duality asserts that the functors
\[ \text{alg}_k \leftrightarrow \text{coalg}_k \]
are adjoint, [5 Thm. 6.0.5]. That is, for any \( k \)-algebra \( A \) and any \( k \)-coalgebra \( C \), there is a natural one-to-one correspondence between the homomorphisms \( \text{alg}_k(A, C^*) = \text{coalg}_k(C, A^*) \).

Moreover, we have [5 Lemma 6.0.1] that for \( f \in \text{alg}_k(A, B) \), the dual map \( f^* \) determines a \( k \)-coalgebra morphism \( f^* \in \text{coalg}_k(B^*, A^*) \).

For a \( k \)-algebra \( A \) one can define the contravariant functor \( \text{rep}(A) \) describing its finite dimensional representations [3 Example 2.1.9]
\[ \text{rep}(A) : \text{coalg}_{k}^{fd} \rightarrow \text{sets} \quad C \mapsto \text{alg}_k(A, C^*) \]
from finite dimensional \( k \)-coalgebras \( \text{coalg}_{k}^{fd} \) to \( \text{sets} \), which commutes with finite direct limits. As on finite dimensional \( k \)-(co)algebras Kostant duality is an anti-equivalence of categories.

we might as well describe \( \text{rep}(A) \) as the noncommutative thin scheme represented by \( A^o \)
\[ \text{rep}(A) : \text{alg}_{k}^{fd} \rightarrow \text{sets} \quad B = C^* \mapsto \text{alg}_k(A, B = C^*) = \text{coalg}_k(C = B^*, A^*) \]
the latter equality follows again from Kostant duality. Therefore, we propose

**Definition 1.** The noncommutative affine scheme \( \text{rep}(A) \) is the noncommutative (thin) scheme corresponding to the dual \( k \)-coalgebra \( A^o \) of \( A \).

2. \( \text{simp}(A) = \text{corad}(A^o) \)

The dual \( k \)-coalgebra \( A^o \) is usually a huge object and hence contains a lot of information about the \( k \)-algebra \( A \). Let us begin by recalling how the geometry of a commutative affine \( k \)-scheme \( X \) is contained in the dual coalgebra \( A^o \) of its coordinate ring \( A = \mathbb{k}[X] \).

Recall that a coalgebra \( D \) is said to be simple if it has no proper nontrivial subcoalgebras. In particular, a simple coalgebra \( D \) is finite dimensional over \( k \) and by duality is such that \( D^* \) is a simple \( k \)-algebra, that is, \( D^* \) is a central simple \( L \)-algebra where \( L \) is a finite separable extension of \( k \).

Hence, in case \( A = \mathbb{k}[X] \) (and \( k \) is separably closed) we have that all simple subcoalgebras of \( A^o \) are one-dimensional (and hence are spanned by a group-like element), because they correspond to simple representations of \( A \).

That is, \( A^o \) is pointed and by [5 Prop. 8.0.7] we know that any cocommutative pointed coalgebra is the direct sum of its pointed irreducible components (at the algebra level, this says that a semi-local commutative algebra is the direct sum of locals). Therefore,
\[ A^o = \bigoplus_{x \in X} C_x \]
where each \( C_x \) is pointed irreducible and cocommutative. As such, each \( C_x \) is a subcoalgebra of the enveloping coalgebra of the abelian Lie algebra on the tangent space \( T_x(X) \).

That is, we recover the points of \( X \) as well as tangent information from the dual coalgebra \( A^o \).

But then, the dual algebra of \( A^o \), that is the 'noncommutative' algebra of functions \( A^{o*} \) decomposes as
\[ A^{o*} = \bigoplus_{x \in X} \mathcal{O}_{x,X} \]
the direct sum of the completions of the local algebras at points. The diagonal embedding \( A = \mathbb{k}[X] \hookrightarrow A^{o*} \) inevitably leads to the structure sheaf \( \mathcal{O}_X \).

We will now associate a topological space associated to any \( k \)-algebra \( A \), generalizing the space of points equipped with the Zariski topology when \( A \) is a commutative affine
\(k\)-algebra. In the next section we will describe the dual coalgebra \(A^o\) when \(A\) is a noncommutative affine \(k\)-algebra.

The coradical \(\text{corad}(C)\) of a \(k\)-coalgebra \(C\) is the (direct) sum of all simple subcoalgebras of \(C\). It is also the direct sum of all simple subcomodules of \(C\), when \(C\) is viewed as a left (or right) \(C\)-comodule.

In the example above, when \(A = k[X]\), we have that \(\text{corad}(A^o) = \bigoplus_{x \in X} k\ ev_x\) where the group-like element \(ev_x\) is evaluation in the point \(x\). This motivates:

**Definition 2.** For a \(k\)-algebra \(A\) we define the space of points \(\text{simp}(A)\) to be the set of direct summands of \(\text{corad}(\text{rep}(A)) = \text{corad}(A^o)\). That is, \(\text{simp}(A)\) is the set of simple subcoalgebras of \(\text{rep}(A)\).

By Kostant duality it follows that \(\text{simp}(A)\) is the set of all finite dimensional simple algebra quotients of the \(k\)-algebra \(A\), or equivalently, the set of all isomorphism classes of finite dimensional simple \(A\)-representations, explaining the notation.

We can equip this set with a Zariski topology in the usual way, using the evaluation map

\[
A^o \times A \xrightarrow{ev} k \quad (f, a) \mapsto f(a)
\]

when restricted to the subcoalgebra \(\text{corad}(A^o)\). Note that the evaluation map actually defines a measuring of \(A\) to \(k\) [5, Prop. 7.0.3], that is, \(A^o \otimes A \xrightarrow{ev} k\) satisfies

\[
ev(f \otimes aa') = \sum_{(f)} f_1(a)f_2(a') \quad \text{and} \quad ev(f \otimes 1) = \epsilon(f)1_k
\]

**Definition 3.** The Zariski topology of a \(k\)-algebra \(A\) is the set \(\text{simp}(A)\) equipped with the topology generated by the basic closed sets

\[\mathbb{V}(a) = \{ S \in \text{simp}(A) \mid ev(S \otimes a) = 0, \text{ that is } f(a) = 0, \forall f \in S \}\]

Having associated a topological space to a \(k\)-algebra, one might ask when this is a functor. Functoriality has always been a problem in noncommutative geometry. Indeed, a simple \(B\)-representation does not have to remain a simple \(A\)-representation under restriction of scalars via \(\phi : A \longrightarrow B\).

Still, if we define \(\text{rep}(A) = A^o\), we get functionality for free. If \(A \xrightarrow{\phi} B\) is an algebra morphism, we have seen that the dual map maps \(B^o\) to \(A^o\), so we have a morphism

\[B^o = \text{rep}(B) \xrightarrow{\phi^*} \text{rep}(A) = A^o\]

A coalgebra is the direct limit of its finite dimensional coalgebras, and they correspond under duality to finite dimensional algebras. Hence, \(\phi^*\) is the natural map on finite dimensional representations by restriction of scalars.

The observed failure of functoriality on the level of points translates on the coalgebra-level to the fact that for a coalgebra map \(B^o \longrightarrow A^o\) the coradical \(\text{corad}(B^o)\) does not have to be mapped to \(\text{corad}(A^o)\), in general.

However, when \(\text{corad}(B^o)\) is cocommutative, we do have that \(\phi^*(\text{corad}(B^o)) \subset \text{corad}(A^o)\) by [5, Thm. 9.1.4]. In particular, we recover the functor of points in commutative algebraic geometry.

Clearly, we still have \(\text{corad}(B^o) \longrightarrow A^o\) in general. This corresponds to the fact that there is always a map \(\text{simp}(B) \longrightarrow \text{rep}(A)\).

Next, let us turn to the algebra of functions on \(\text{rep}(A)\). By definition we have

\[k[\text{rep}(A)] = A^{oo}\]

and we can ask how this algebra relates to the algebra \(A\).

In general, it is not true that \(A \longrightarrow A^{oo}\). This only holds when \(A\) is dense in \(A^o\) in which case the \(k\)-algebra is said to be proper, see [5, §6.1].

In the commutative case, when \(A\) is a finitely generated \(k\)-algebra, then \(A\) is indeed proper and this is a consequence of the Hilbert Nullstellensatz and the Krull intersection theorem.
When $A$ is noncommutative, this is no longer the case. For example, if $A = A_n(k)$ the Weyl algebra over a field of characteristic zero $k$, then $A$ is simple whence has no two-sided ideals of finite codimension. As a result $A^o = 0$! As our proposal for the noncommutative affine scheme $\mathrm{rep}(A)$ is based on finite dimensional representations of $A$, it will not be suitable for $k$-algebras having few such representations.

3. THE DUAL COALGEBRA $A^o$

In general though, $A^o$ is a huge object, so it is very difficult to describe explicitly. In this section, we will begin to tame $A^o$ even when $A$ is noncommutative.

In order not to add extra problems, we will assume that $k$ is separably closed in this section. The general case can be recovered by taking $\text{Gal}(k/k)$-invariants (replacing quivers by species in the sequel).

Over a separably closed field $k$ all simple subcoalgebras are full matrix coalgebras $M_n(k)^*$, that is, $M_n(k)^* = \oplus_{ij} k e_{ij}$ with $\Delta(e_{ij}) = \sum_{k=1}^n e_{ik} \otimes e_{kj}$ and $\epsilon(e_{ij}) = \delta_{ij}$.

Hence, $\text{corad}(A^o) = \oplus_M M_{n_M}(k)^*$ where the sum is taken over all finite dimensional simple $A$-representations $M$, each having dimension $n_M$.

In algebra, one can resize idempotents by Morita-theory and hence replace full matrices by the basefield. In coalgebra-theory there is an analogous duality known as Takeuchi equivalence, see [6].

The isotypical decomposition of $\text{corad}(A^o)$ as an $A^o$-comodule is of the form $\oplus S C_S^\otimes n_M$, the sum again taken over all simple $A$-representations. Take the $A^o$-comodule $E = \oplus S C_S$ and its coendomorphism coalgebra

$$A^\dagger = \text{coend}^A(E)$$

then Takeuchi-equivalence (see for example [1, §4, §5] and the references contained in this paper for more details) asserts that $A^o$ is Takeuchi-equivalent to the coalgebra $A^\dagger$ which is pointed, that is, $\text{corad}(A^\dagger) = k \text{simp}(A) = \oplus S k g_S$ with one group-like element $g_S$ for every simple $A$-representation. Remains to describe the structure of the full basic coalgebra $A^\dagger$.

For a (possibly infinite) quiver $\bar{Q}$ we define the path coalgebra $\mathbb{k}\bar{Q}$ to be the vectorspace $\oplus_{p \in \mathbb{k}p}$ with basis all oriented paths $p$ in the quiver $\bar{Q}$ (including those of length zero, corresponding to the vertices) and with structural maps induced by

$$\Delta(p) = \sum_{p'p''} p' \otimes p'' \quad \text{and} \quad \epsilon(p) = \delta_{l(p)}$$

where $p'p''$ denotes the concatenation of the oriented paths $p'$ and $p''$ and where $l(p)$ denotes the length of the path $p$. Hence, every vertex $v$ is a group-like element and for an arrow $\bar{a} \rightarrow \bar{b}$ we have $\Delta(a) = v \otimes a + a \otimes w$ and $\epsilon(a) = 0$, that is, arrows are skew-primitive elements.

For every natural number $i$, we define the $\text{ext}^i$-quiver $\text{ext}^i_A$ to have one vertex $v_S$ for every $S \in \text{simp}(A)$ and such that the number of arrows from $v_S$ to $v_T$ is equal to the dimension of the space $\text{Ext}^i_A(S, T)$. With $\text{ext}_A^i$ we denote the $k$-vectorspace on the arrows of $\text{ext}_A^i$.

The Yoneda-space $\text{ext}_A^1 = \oplus \text{ext}_A^i$ is endowed with a natural $A_\infty$-structure [2], defining a linear map (the homotopy Maurer-Cartan map, [4])

$$\mu = \oplus_i m_i : \text{ext}_A^1 \longrightarrow \text{ext}_A^2$$

from the path coalgebra $k\text{ext}_A^1$ of the $\text{ext}^1$-quiver to the vectorspace $\text{ext}_A^2$, see [2] §2.2 and [4].

**Theorem 1.** The dual coalgebra $A^o$ is Takeuchi-equivalent to the pointed coalgebra $A^\dagger$ which is the sum of all subcoalgebras contained in the kernel of the linear map

$$\mu = \oplus_i m_i : \text{ext}_A^1 \longrightarrow \text{ext}_A^2$$

determined by the $A_\infty$-structure on the Yoneda-space $\text{ext}_A^1$. 
We can reduce to finite subquivers as any subcoalgebra is the limit of finite dimensional subcoalgebras and because any finite dimensional $A$-representation involves only finitely many simples. Hence, the statement is a global version of the result on finite dimensional algebras due to B. Keller [2, §2.2].

Alternatively, we can use the results of E. Segal [4]. Let $S_1, \ldots, S_r$ be distinct simple finite dimensional $A$-representations and consider the semi-simple module $M = S_1 \oplus \ldots \oplus S_r$ which determines an algebra epimorphism

$$
\pi_M : A \longrightarrow M_{a_1}(k) \oplus \ldots \oplus M_{a_n}(k) = B
$$

If $m = Ker(\pi_M)$, then the $m$-adic completion $\hat{A}_m = \varprojlim A/m^n$ is an augmented $B$-algebra and we are done if we can describe its finite dimensional (nilpotent) representations. Again, consider the $A_\infty$-structure on the Yoneda-algebra $Ext^*_A(M, M)$ and the quiver on $r$-vertices $\text{Ext}^1_A(M, M)$ and the homotopy Mauer-Cartan map

$$
\mu_M = \oplus_i m_i : \text{Ext}^1_A(M, M) \longrightarrow \text{Ext}^2_A(M, M)
$$

Dualizing we get a subspace $\text{Im}(\mu_M^*)$ in the path-algebra $\text{Ext}^1_A(M, M)^*$ of the dual quiver. Ed Segal’s main result [4, Thm 2.12] now asserts that $\hat{A}_m$ is Morita-equivalent to

$$
\hat{A}_m \cong \frac{(\text{Ext}^1_A(M, M)^*)}{\text{Im}(\mu_M^*)}
$$

where $(\text{Ext}^1_A(M, M)^*)$ is the completion of the path-algebra at the ideals generated by the paths of positive length. The statement above is the dual coalgebra version of this.

REFERENCES


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