A SHEAF OF HOCHSCHILD COMPLEXES ON QUASI-COMPACT OPENS

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ABSTRACT. For a scheme $X$, we construct a sheaf $C$ of complexes on $X$ such that for every quasi-compact open $U \subseteq X$, $C(U)$ is quasi-isomorphic to the Hochschild complex of $U$ (Lowen and Van den Bergh, 2005). Since $C$ is moreover acyclic for taking sections on quasi-compact opens, we obtain a local to global spectral sequence for Hochschild cohomology if $X$ is quasi-compact.

1. Introduction

Let $X$ be a scheme over a field $k$. In [11], the Hochschild complex $C(X, \mathcal{O}_X)$ of $X$ is defined to be the Hochschild complex of the abelian category $\text{Mod}(X)$ of sheaves on $X$. Its cohomology theory coincides with various notions of Hochschild cohomology of $X$ considered in the literature, for example by Swan [14] and Kontsevich [8], which in the commutative case agree with the earlier theory of Gerstenhaber-Schack [2].

For a basis $b$ of affine opens of $X$, there is an associated $k$-linear category (also denoted by $b$) and there is a quasi-isomorphism

$$C(X, \mathcal{O}_X) \cong C(b)$$

where $C(b)$ is the Hochschild complex of the $k$-linear category $b$ ([2,1]). The Hochschild complexes have a considerable amount of extra structure containing in particular the cup-product and the Gerstenhaber bracket. This extra structure is important for deformation theory. It is captured by saying that the complexes are $B_\infty$-algebras [3,10], and $\cong$ as above means the existence of an isomorphism in the homotopy category of $B_\infty$-algebras. Let $\mathcal{O}_b$ be the restriction of $\mathcal{O}_X$ to the basis $b$. Then $\cong$ above is reflected in the fact that there is an equivalence between the deformation theory of $\text{Mod}(X)$ as an abelian category [12] and the deformation theory of $\mathcal{O}_b$ as a twisted presheaf [9].

If we consider the restrictions $b|_U$ of $b$ to open subsets $U \subseteq X$, we obtain a presheaf of Hochschild complexes on $X$:

$$C_b : U \mapsto C(b|_U).$$
To relate the “global” Hochschild complex $C(b)$ to the “local” Hochschild complexes $C(b|U)$ of certain open subsets $U \subset X$, it would be desirable for $C_b$ to be a sheaf, which is preferably acyclic for taking global sections. Unfortunately, $C_b$ is not even a separated presheaf with regard to finite coverings. In this paper, we construct a sheaf $C$ of $B_\infty$-algebras such that

1. $C(U, \mathcal{O}_U) \cong C(U)$ for $U$ quasi-compact open.
2. $C$ is acyclic for taking quasi-compact sections, i.e. $R\Gamma(U, C) = C(U)$ for $U$ quasi-compact open.

For $U$ quasi-compact open, $C(U)$ is obtained as a colimit of complexes $C(b|U)$ over a collection $\mathcal{B}(U)$ of bases of $U$ ($\mathcal{B}$). The properties of $C$ depend upon the choice of a good presheaf $B$ of bases (Definition 2.1).

From properties (1) and (2), we readily deduce the existence of a local to global spectral sequence

$$E_2^{p,q} = H^p(X, H^qC) \Rightarrow H^{p+q}C(X)$$

for Hochschild cohomology for a quasi-compact scheme $X$ (Theorem 4.1).

We should remark that for a smooth separated scheme, another sheaf of $B_\infty$-algebras $D_{\text{poly}}$ is considered, for example, by Kontsevich [7], Van den Bergh [15], and Yekutieli [16]. Let $C(\mathcal{O}(U))$ be the Hochschild complex of the ring $\mathcal{O}(U)$, and let $C_{\text{poly}}(\mathcal{O}(U))$ be the subcomplex which consists of the polydifferential operators, i.e. multilinear maps $\mathcal{O}(U)^{\otimes p} \rightarrow \mathcal{O}(U)$ which are differential operators in each argument. Then for $U$ affine open, $D_{\text{poly}}$ satisfies

$$D_{\text{poly}}(U) \cong C_{\text{poly}}(\mathcal{O}(U)).$$

The complex $R\Gamma(X, D_{\text{poly}})$ computes the Hochschild cohomology of $X$, but a priori does not inherit the structure of a $B_\infty$-algebra. One way to overcome this problem is by using a fibrant resolution $D_{\text{poly}} \rightarrow F_{\text{poly}}$ in the model category of presheaves of $B_\infty$-algebras as defined by Hinich [5]. Alternatively, in [15, Appendix B], Van den Bergh constructs a quasi-isomorphic object $R\Gamma(X, D_{\text{poly}})^\text{tot}$ that does inherit this structure (the construction, which uses pro-hypercoverings, is functorial and inherits any operad-algebra structure). Moreover, $R\Gamma(X, D_{\text{poly}})^\text{tot}$ is isomorphic to $C(X, \mathcal{O}_X)$ in the homotopy category of $B_\infty$-algebras [15 Theorem 3.1, Appendices A, B] and by [15 Appendix B.10], we actually have $R\Gamma(X, D_{\text{poly}})^\text{tot} \cong \Gamma(X, F_{\text{poly}})$ in the same sense.

Finally, as to the existence of a local to global spectral sequence for Hochschild cohomology for a general ringed space $(X, \mathcal{O}_X)$, a proof using hypercoverings is in preparation [10].

## 2. Presheaves of Hochschild complexes

### 2.1. The Hochschild complex of a scheme.

Throughout, $k$ is a field. Let $(X, \mathcal{O}_X)$ be a scheme over $k$ and let $b$ be a basis of affine opens. We use the same notation for the associated $k$-linear category with $b$ as objects and

$$b(V, U) = \begin{cases} \mathcal{O}_X(V) & \text{if } V \subset U, \\ 0 & \text{else.} \end{cases}$$

In [11 §7.1], the Hochschild complex $C(X, \mathcal{O}_X)$ of $X$ is defined, and in [11 Theorem 7.3.1], there is shown to be a quasi-isomorphism

$$C(X, \mathcal{O}_X) \cong C(b),$$
where $C(b)$ is the Hochschild complex of the $k$-linear category $b$ \[13\], i.e.

$$C^p(b) = \prod_{U_0, \ldots, U_p \in b} \text{Hom}_k(b(U_{p-1}, U_p) \otimes_k \cdots \otimes_k b(U_0, U_1), b(U_0, U_p)),$$

and the differential is the usual Hochschild differential. More concretely, we have

$$C^p(b) = \prod_{U_0 \subseteq U_1 \subseteq \cdots \subseteq U_p \in b} \text{Hom}_k(O_X(U_{p-1}) \otimes_k \cdots \otimes_k O_X(U_0), O_X(U_0)),$$

and

$$C^0(b) = \prod_{U_0 \in b} O_X(U_0).$$

Hence this complex combines the nerve of the poset $b$ with the algebraic structure of $O_X$. In fact, both complexes are $B_{\infty}$-algebras $[3, 6]$, and $\cong$ means the existence of an isomorphism in the homotopy category of $B_{\infty}$-algebras.

2.2. The presheaf $C_B$ of Hochschild complexes. For an arbitrary open subset $U \subset X$, put $b_U = \{B \in b \mid B \subset U\}$. Then $b|_U$ is a basis of affine opens for the scheme $(U, O_U)$; hence we have a quasi-isomorphism $C(U, O_U) \cong C(b|_U)$.

For $V \subset U$ there is an obvious restriction map $C(b|_U) \rightarrow C(b|_V)$.

We thus obtain a presheaf

$$C_b : U \mapsto C_b(U) = C(b|_U)$$

of Hochschild complexes on $X$. It is readily seen that in general, $C_b$ fails to be a sheaf. Indeed, suppose we have $W \in b$ and $W = U \cup V$ with $U$ and $V$ proper open subsets. Then there is a non-zero element $\varphi = (\varphi_{U_0})_{U_0} \in C^0_b(W)$ with

$$\varphi_{U_0} = \begin{cases} 1 \in O_X(U_0) & \text{if } U_0 = W, \\ 0 & \text{else}, \end{cases}$$

whose restriction to $U$ and $V$ is zero. In this example, the fact that $W = U \cup V$ makes the presence of $W$ in $b$ redundant. This suggests that to obtain a sheaf, we must work with variable bases, as will be done in the next section.

2.3. The presheaf $C_B$ of colimit Hochschild complexes. In this section instead of considering $C_b(U)$ for a fixed basis $b$ of $X$, we will consider a colimit of complexes $C(b)$ over different bases $b$ of $U$. More precisely, we are looking for collections $B(U)$ of bases of affine opens of $U$, which allow us to define “colimit Hochschild complexes”

$$C_B(U) = \text{colim}_{b \in B(U)} C(b).$$

Here $B(U)$ is ordered by $\supset$ and $b \supset b'$ corresponds to the canonical $C(b) \rightarrow C(b')$.

Since we do not want the colimit to change the cohomology, we want it to be a filtered colimit. In particular, this is the case if $B(U)$ is closed under intersections, i.e. if we have the operation

$$B(U) \times B(U) \longrightarrow B(U) : (b, b') \mapsto b \cap b'.$$

Note that in general, $b \cap b'$ need not even be a basis. If $B(U) \neq \varnothing$ and we have \[1\], then there are quasi-isomorphisms $C(U, O_U) \cong C_B(U).$
For \( C_B : U \mapsto C_B(U) \) to become a presheaf, we need restriction operations

\[
(2) \quad B(U) \longrightarrow B(V) : b \mapsto b|_V = \{ B \in b \mid B \subset V \}
\]

for \( V \subset U \), making \( B \) itself into a presheaf of collections of bases. In this way, \( C_B \) clearly becomes a presheaf of \( B_{\infty} \)-algebras on \( X \).

In order to prove Proposition 3.1 in the next section, we need two more operations on \( B \). First, we want to take the union of bases coinciding on the intersection of their domains; i.e. we want the operation

\[
(3) \quad B(U) \times_{B(U \cap V)} B(V) \longrightarrow B(U \cup V) : (b, b') \mapsto b \cup b'.
\]

Second, we want to refine bases by plugging in finer bases; i.e. for \( V \subset U \) we want the operation

\[
(4) \quad B(U) \times B(V) \longrightarrow B(U) : (b_U, b_V) \mapsto b_U \circ b_V = \{ B \in b_U \mid B \subset V \implies B \in b_V \}.
\]

Note that (1) is just a special case of (4). Also, combining (2), (3) and (4) yields the following refinement operation on \( B \). If \( \delta \) is any finite collection of open subsets of \( U \) (not necessarily covering \( U \)), we have

\[
(5) \quad B(U) \longrightarrow B(U) : b \mapsto b_{\delta} = \{ B \in b \mid V \subset \delta \implies \exists D \in \delta, V \subset D \}.
\]

**Definition 2.1.** \( B \) is called good if \( B(X) \neq \emptyset \) and \( B \) has the operations (1),..., (5).

We will now show that there exists a good \( B \).

**Proposition 2.2.**

1. If \( B \) with \( B(X) \neq \emptyset \) has (2), (3) and (4), then it is good.
2. Let \( b \) be any basis of affine opens of \( X \). There exists a smallest good \( B \) with \( b \in B(X) \). This \( B \) is given by

\[
B(U) = \{(b|_U)_{\delta_1},...,\delta_n \mid \delta_1 \subset \text{open}(U), |\delta_1| < \infty \}.
\]

**Proof.** (1) follows from the discussion above. For (2), first note that \( B \) is obviously contained in any good \( B' \). If \( V \subset U \) and \( \delta \) is a collection in \( U \), we put \( \delta|_V = \{ D \cap V \mid D \in \delta \} \). For any basis \( b' \) of \( U \), we have \((b'|_V)_{\delta|_V} = (b'|_V)_{\delta|_V} \) holds. For (1), note that \((b|_U)_{\delta_1},...,\delta_n \circ (b|_V)_{\epsilon_1},...,\epsilon_m = (b|_U)_{\delta_1},...,\delta_n,\epsilon_1,...,\epsilon_m \). Finally for (3), if \((b|_U)_{\delta_1},...,\delta_n \) and \((b|_V)_{\epsilon_1},...,\epsilon_m \) coincide on \( U \cap V \), then their union equals \((b|_{U \cup V})_{\delta_1},...,\delta_n,\epsilon_1,...,\epsilon_m,\{U,V\}) \). \( \square \)

### 3. Sheaves of Hochschild complexes

3.1. **The presheaf \( C_B \) for a good \( B \).** From now on, \( B \) is a good presheaf of bases, and we consider the presheaf \( C_B \) of colimit Hochschild complexes as defined in (2.3).

**Proposition 3.1.**

1. \( C_B \) is flabby.
2. \( C_B \) satisfies the sheaf condition with respect to finite coverings.

**Proof.** (2) By induction, we may consider \( U = U_1 \cup U_2 \) and the given elements \( \varphi_i \in C_B(U_i) \) such that \( \varphi_i|_{U_{12}} = \varphi_2|_{U_{12}} \), where \( U_{12} = U_1 \cap U_2 \). Let \( \varphi_1 \) be a representing element in \( C^p(b_1) \) for a basis \( b_1 \in B(U_1) \) and let \( b' \subset b_1|_{U_{12}} \cap b_2|_{U_{12}} \) be a basis in \( B(U_{12}) \) for which \( \varphi_1|_{U_{12}} = \varphi_2|_{U_{12}} \) coincide in \( C^p(b') \). Put \( b'_1 = b_1 \circ b' \in B(U_1) \) (using (1)) and put \( b = b'_1 \cup b'_2 \in B(U \cup V) \) (using (3)). We can now easily give an element \( \varphi \in C^p(b) \), which represents a glueing of \( \varphi_1 \) and \( \varphi_2 \) on \( U \), by specifying its
value for $V_0 \subset \cdots \subset V_p$; if $V_p \in b'_i$, we use the element specified by $\varphi_i$. This is well
defined since $V_p \in b'_i \cap b'_2$ implies $V_p \in b'$. It is a gluing of the $\varphi_i$ since $\varphi$ and $\varphi_i$
coincide on $b'_i \subset b_i$.

Now suppose we have an element $\varphi \in \mathcal{C}^p(b')$ for $b' \in \mathcal{B}(U)$ and suppose we
have bases $b_i \subset b'_i|V_i$ for which $\varphi|_{U_i}$ becomes zero in $\mathcal{C}^p(b_i)$. If we put $b'_i = b_i \circ (b_1|V_{12} \cap b_2|V_{12})$ and $b = b'_1 \cup b'_2$, then $\varphi$ becomes zero in $\mathcal{C}^p(b')$.

(1) Consider the restriction map $\mathcal{C}^p(U) \to \mathcal{C}^p(V)$ for $V \subset U$. If $\varphi \in \mathcal{C}^p(b)$
is a representing element in the codomain, we can lift it to $\bar{\varphi} \in \mathcal{C}^p(b' \circ b)$, where $b' \in \mathcal{B}(U)$ is arbitrary and the value of $\bar{\varphi}$ for $V_0 \subset \cdots \subset V_p$ is the value specified by
$\varphi$ if $V_p \subset V$ and is zero otherwise. □

3.2. The sheaf $\mathcal{C}_{qc}$ of colimit Hochschild complexes. Let $\mathcal{C}(X) \subset \text{open}(X)$
be the subposet of quasi-compact opens with the induced Grothendieck topology. We
immediately get:

**Proposition 3.2.** The restriction $\mathcal{C}_{qc}$ of $\mathcal{C}$ to $\mathcal{C}(X)$ is a sheaf.

**Proof.** Since every covering of a quasi-compact $U \subset X$ has a finite subcovering,
it suffices to check the sheaf condition on finite coverings, which is done in Proposition 3.1. □

3.3. The sheaf $\mathcal{C} = \mathcal{C}_G$. Let $\text{Pr}(X)$ and $\text{Sh}(X)$ (resp. $\text{Pr}_{qc}(X)$ and $\text{Sh}_{qc}(X)$) be
the categories of presheaves and sheaves on $X$ (resp. on $\mathcal{C}(X)$). Since $\mathcal{C}(X)$ is a
category of $X$, by the (proof of the) Lemme de Comparaison [1], there is a commutative square

\[
\begin{array}{ccc}
\text{Pr}(X) & \xrightarrow{(-)_{qc}} & \text{Pr}_{qc}(X) \\
\alpha \downarrow & & \downarrow \alpha' \\
\text{Sh}(X) & \xrightarrow{(-)_{qc}} & \text{Sh}_{qc}(X)
\end{array}
\]

in which the vertical arrows are sheafifications, the horizontal arrows are restrictions
to $\mathcal{C}(X)$, and the lower horizontal arrow is an equivalence. Let $\mathcal{C} = a_!\mathcal{C}_G$ be the
sheafification of $\mathcal{C}_G$.

**Proposition 3.3.** If $U \subset X$ is a quasi-compact open, then

\[\mathcal{C}_G(U) \to \mathcal{C}(U)\]

is an isomorphism. In particular, there is a quasi-isomorphism

\[\mathcal{C}(U, \mathcal{O}_U) \cong \mathcal{C}(U)\]

**Proof.** By Proposition 3.2 we have $\mathcal{C}_{qc} \cong a_! \mathcal{C}_{qc} \cong (a_! \mathcal{C}_G)_{qc}$. □

**Proposition 3.4.** $\mathcal{C}^p$ is acyclic for taking quasi-compact sections; i.e. for $U \subset X$
a quasi-compact open and $i > 0$, we have $H^i(U, \mathcal{C}^p) = 0$.

**Proof.** By Propositions 3.1 and 3.3 the restriction maps $\mathcal{C}^p(X) \to \mathcal{C}^p(U)$ are
surjective for $U$ quasi-compact. The rest of the proof is along the lines of the
classical proof that flabby sheaves are acyclic for taking global sections. □
4. LOCAL TO GLOBAL SPECTRAL SEQUENCE

In this section, \( X \) is a quasi-compact scheme and \( \mathcal{C} \) is the sheaf of complexes of \( \mathcal{O}_X \). In particular, there are quasi-isomorphisms \( \mathcal{C}(U, \mathcal{O}_U) \cong \mathcal{C}(U) \) for \( U \) quasi-compact open. We obtain a local to global spectral sequence for Hochschild cohomology:

**Theorem 4.1.** There is a local to global spectral sequence

\[
E_2^{p,q} = H^p(X, H^q \mathcal{C}) \Rightarrow H^{p+q}(\mathcal{C}(X)).
\]

**Proof.** Since, by Proposition 3.4, \( \mathcal{C} \) is a bounded below complex of acyclic objects for \( \Gamma \), \( \mathcal{C} \) is itself acyclic, i.e. \( R\Gamma(X, \mathcal{C}) = \mathcal{C}(X) \). So the above is just the hypercohomology spectral sequence \([4, 2.4.2]\) for the complex of sheaves \( \mathcal{C} \). \( \square \)

**References**


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